Inductance Calculation Techniques --- Part II: Approximations and Handbook Methods

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INDEX OF SYMBOLS

A	Area enclosed by coil
a	Coil mean radius
b	Coil axial thickness
c	Coil radial thickness
D	Side length of square coil
d	Wire-to-wire spacing
K	Nagoaka coil constant
l	Coil length
p	Perimeter enclosed by coil
P, F	Grover disk coil constants
R	Circular wire radius
s	Side length of generic polygon
w	Trace width
x, y	Mean side lengths of rectangular coil
μ_o	Magnetic permeability of free space $4\pi\times10^{-7}$ H/m

I. INTRODUCTION

In the second part of this two part series on inductance calculation techniques, approximation techniques and handbook methods are shown for air-core structures that do not easily lend themselves to closed-form solutions. A set of references is also given which is useful for finding the inductance of many different loop shapes. Included are inductance calculations for polygons, disk coils, finite-length solenoids and flat planar spirals. In some cases results are given without significant explanation, as the calculations are very complicated and the full calculations may be found in the references given. Many of the older references work out inductance problems in English units; to ease design these results have been converted to MKS units with inductance in Henries and length scales in meters.

II. REFERENCE REVIEW

Inductance calculation references necessarily start with Maxwell's seminal work [1], first published in 1873. Maxwell worked out some interesting inductance problems, including finding the mutual inductance between circular coaxial filaments [1, pp. 339], and finding the size and shape of a coil which maximizes inductance for a given length of wire [1, pp. 345].

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A seminal reference for inductance calculation (which, unfortunately is now out of print) is the Frederick Grover book [2]. Professor Grover spent most of his professional life calculating inductance of different kinds of wire loops, and many useful examples and tables are given for loops of interesting shapes and sizes. The reference is especially useful as it gives correction factors to account for high frequency operation and other effects such as the insulation thickness of wires. He was also a consultant for the U.S. Bureau of Standards, and the results of some of his work are found in [3].

The Wheeler references [4,5] give simple results for circular coils, and are particularly useful where analytic expressions are needed. The empirically-derived analytic expressions in [5] are remarkably accurate for multiple-turn, circular coils.

Another good reference is the *Radio Engineers' Handbook*, edited by Terman [6]. This reference shows the inductance of many different coil shapes such as single-turn loops, rectangles, multiple-layer coils, and solenoids of various lengths. The Dwight reference [7] is very useful for calculating high-frequency effects in inductors, and covers skin and proximity effects. The Roters reference [8] shows a useful method by which leakage inductance in transformers can be estimated. The Smythe reference [9] is a great work covering classical field and inductance calculations. *Solenoid Magnet Design* [10] by D. Bruce Montgomery, covers design aspects of solenoid-shaped coils from a superconducting magnet point of view. Thermal aspects of inductor design are well covered here.

Other references, as needed, are given in the text. A more comprehensive bibliography of inductance calculation references is given on the author's business website, given in the REFERENCES section.

III. APPROXIMATE AND CLOSED-FORM RESULTS FOR COMMON STRUCTURES

1. Circular Wire Loop

The self-inductance of a straight conductor of length l and radius R, neglecting the effects of nearby conductors (i.e. assuming that the return current is far away) is given by [2, pp. 35]:

$$L \approx \frac{\mu_o l}{2\pi}\left[\ln\left(\frac{2l}{R}\right) - \frac{3}{4}\right] \qquad [1]$$

Many other magnetic structures can be modeled as simpler structure, since inductance is in general a weak function of loop shape for loops with loop radius much larger than wire cross section.

Surprisingly, there is no closed-form solution for the inductance of a filamentary loop (since the expression for inductance blows up if the wire radius goes to zero). A circular loop of round wire (**Figure 1**) with loop radius a and wire radius R has the approximate low frequency inductance [2, pp. 143], [4]:

$$L = \mu_o a\left[\ln\left(\frac{8a}{R}\right) - 1.75\right] \qquad [2]$$

Using this formula, the inductance of a 1 meter circumference loop of 14 gauge wire is 1.12 μH; for 16 gauge wire it's 1.17 μH; and for 18 gauge wire it's 1.21 μH. Note the weak dependence of inductance on wire diameter, due to the natural log in the expression.

A crude approximation for circular loops is $L \approx \mu_o \pi a$ [11, pp. 56] which predicts an inductance of 0.63 μH for a 1 meter circumference wire loop.

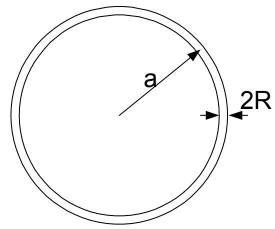


Figure 1. Circular wire loop

2. Parallel-wire line

For two parallel wires whose length l is great compared to their distance d apart, (**Figure 2**) the inductance of the loop is [2, pp. 39]:

$$L = \frac{\mu_o l}{\pi} \ln\left[\frac{d}{R} + \frac{1}{4} - \frac{d}{l}\right] \qquad [3]$$

For $l = 0.5$ meter and a wire-wire spacing $d = 1$ cm, results are: $L = 0.505$ μH for 14 gauge; $L = 0.551$ μH for 16 gauge and $L = 0.598$ μH for 18 gauge. Therefore, for the parallel-wire line with closely-spaced conductors, the inductance is approximately 0.5 microHenries per meter of total wire length.

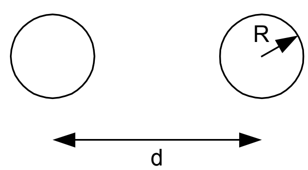


Figure 2. Parallel-wire line

3. Square loop

The self inductance of a square coil made of rectangular wire (**Figure 3**), with depth b (into the paper) small compared to side length D and trace width $2w$ is a complicated expression found in the Zahn reference [12, pp. 343]. However, for $w << D$ a relatively nasty expression can be approximated by:

$$L \approx \frac{2\mu_o D}{\pi}\left[\sinh^{-1}\left(\frac{D}{w}\right) - 1\right] \qquad [4]$$

For instance, a square PC board trace $1\text{cm} \times 1\text{cm}$ with trace width 1 millimeter has an inductance of approximately 16 nanoHenries (assuming that the ground plane is far away, of course).

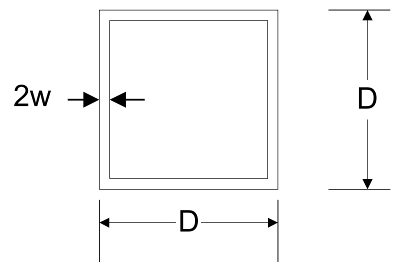


Figure 3. Square coil with rectangular cross section

4. Rectangle of round wire

The inductance of a rectangle of round wire with rectangle side lengths x and y is [2, pp. 60]:

$$L = \frac{\mu_o}{\pi}\left[x\ln\left(\frac{2x}{R}\right) + y\ln\left(\frac{2y}{R}\right) + 2\sqrt{x^2+y^2} - x\sinh^{-1}\left(\frac{x}{y}\right) - y\sinh^{-1}\left(\frac{y}{x}\right) - 1.75(x+y)\right] \qquad [5]$$

5. Polygon of round wire

These results suggest an interesting result for a polygon of wire. The inductance of a generic polygon of wire with perimeter p and area A may be approximated by:

$$L \approx \frac{\mu_o p}{2\pi}\left[\ln\left(\frac{2p}{R}\right) + 0.25 - \ln\left(\frac{p^2}{A}\right)\right] \qquad [6]$$

Note that this function is strongly dependent on the perimeter and weakly dependent on loop area and wire radius. For this reason, the inductance of complicated shapes can often be well approximated by a simpler shape with the same perimeter and/or area.

No closed-form exists to calculate the inductance of a generic polygon of wire. Grover [2, pp. 60] has worked out a variety of cases for polygons with side length s and wire radius R:

Equilateral Triangle
$$L = \frac{3\mu_o s}{2\pi}\left[\ln\left(\frac{s}{R}\right) - 1.15546\right] \qquad [7]$$

Square
$$L = \frac{2\mu_o s}{\pi}\left[\ln\left(\frac{s}{R}\right) - 0.52401\right] \qquad [8]$$

Pentagon
$$L = \frac{5\mu_o s}{2\pi}\left[\ln\left(\frac{s}{R}\right) - 0.15914\right] \qquad [9]$$

Hexagon
$$L = \frac{3\mu_o s}{\pi}\left[\ln\left(\frac{s}{R}\right) + 0.09848\right] \qquad [10]$$

Octagon
$$L = \frac{4\mu_o s}{\pi}\left[\ln\left(\frac{s}{R}\right) - 0.03802\right] \qquad [11]$$

However, it is found in practice [2, pp. 61], [15] that the inductance of a multi-faced polygon may be approximated by replacing a polygon by a simpler plane figure of either equal area of equal perimeter. If the perimeter of the coil is p and the area enclosed is A then the inductance takes the general form:

$$L \approx \frac{\mu_o p}{2\pi}\left[\ln\left(\frac{2p}{R}\right) + 0.25 - \ln\left(\frac{p^2}{A}\right)\right] \qquad [12]$$

This result is approximate, and an empirical fit to the closed-form calculations for polygons above. In general, coils enclosing the same perimeter with similar shapes will have approximately the same inductance. This is a very useful result for strange-shaped coils.

6. Use of filaments

Conductors can be replaced by filaments in order to calculate inductances, often with very accurate results. For straight filaments made of parallel conductors, with length l and filament-filament spacing d (*Figure 4*a), the mutual inductance is [2, pp. 31]:

$$M_{12} = \frac{\mu_o l}{2\pi}\left[\ln\left(\frac{l}{d} + \sqrt{1+\frac{l^2}{d^2}}\right) - \sqrt{1+\frac{d^2}{l^2}} + \frac{d}{l}\right] \qquad [13]$$

Figure 4. Mutual inductance between filaments. (a) Straight conductors (b) Loops

For the coaxial loops, the mutual inductance between loops as found by Maxwell is:

$$M_{12} = \mu_o\sqrt{a_1 a_2}\,\frac{2}{k}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right],\; k = \sqrt{\frac{4a_1a_2}{(a_1+a_2)^2+h^2}} \qquad [14]$$

where $K(k)$ and $E(k)$ are elliptic integrals. Mutual inductance is a very important parameter to calculate, as if the mutual inductance M_{12} is found the force between loops can be found as

$$f_{12} = -I_1 I_2 \nabla M_{12} \qquad [15]$$

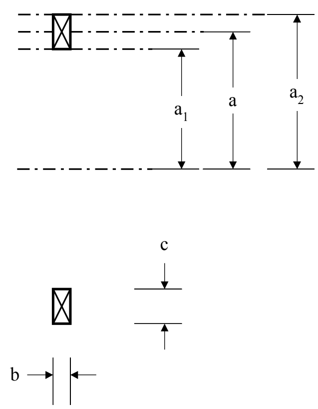
IV. HANDBOOK METHODS

1. Disk coil

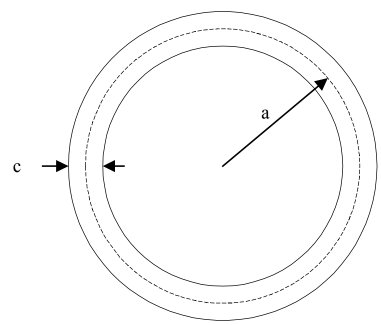
A useful geometry for which tabulated results exist is the round loop with rectangular cross section, with mean radius a, axial thickness b, and trace width c (**Figure 5**). The self-inductance of this single loop is calculated using techniques outlined in Grover [2, pp. 94], where the inductance is shown to be:

$$L = \frac{\mu_o}{4\pi} aPF \qquad [16]$$

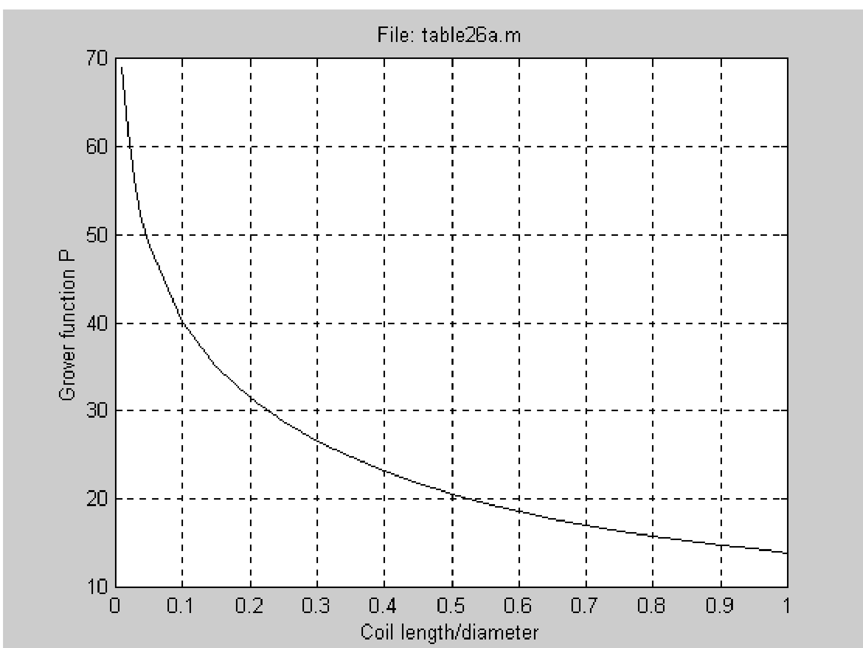
This result is in MKS units, with a in meters and L in Henries. P and F are unitless constants; P is a function of the coil normalized radial thickness $c/2a$ (**Figure 5c**) and applies to a coil of zero axial thickness ($b = 0$), and F accounts for the finite axial length of the coil. For $b << c$ and $c << a$ (coils resembling thin disks) the factor $F \approx 1$, an important limiting case. Therefore, for a thin disk coil with double the mean radius, there is a corresponding doubling of the inductance. If the coil is made of multiple turns of wire, and if c << a the inductance can be approximated by multiplying the above expression by N^2.



(a) Cross sectional view



(b) Top view



(c) Function P, for disk coils

Figure 5. Circular coil with rectangular cross section

An interesting result pops out here .... magnetic scaling laws [13] show that large magnetic elements are more efficient in energy conversion than smaller ones. For this disk coil geometry, this effect can be quantified by considering the ratio of inductance to resistance. The inductance L is approximately proportional to a as shown above. The resistance of the coil is proportional to a/bc, the ratio of current path length to coil cross-sectional area. Therefore, the ratio of inductance to resistance is proportional to bc, or the cross-sectional area of the coil. If all coil lengths are scaled up by the same factor ℓ, this ratio increases by the factor ℓ^2, or the length squared.

2. "Brooks" Coil

An interesting problem is to maximize the inductance with a given length of wire. Maxwell [1, pp. 345] found that the optimal coil has a square cross section with mean diameter 3.7 times the dimension of the square cross section, or $2a = 3.7c$. Brooks and others [14] later refined this estimate and recommend $2a/c = 3$ as the optimum shape, with $b = c$. The result for the Brooks coil is:

$$L = 1.353\mu_o a N^2 \qquad [17]$$

The inductance is a rather weak function of $2a/c$ so the exact geometry isn't so important.

3. Finite-length solenoid

The inductance of a thin-wall finite-length solenoid of radius a and length l made of round wire can be calculated with good accuracy by using one of the Wheeler formulas [4, 11]. If the length l of the solenoid is larger 0.8a, the accuracy is better than 1%:

$$L \approx \frac{10\pi\mu_o N^2 a^2}{9a + 10l} \qquad [18]$$

For the short solenoid with $l < a$, a handbook method may be applied by using the Nagoaka formula [2, pp. 143], [6, pp. 53] where (in MKS units):

$$L = \pi\mu_o a^2 l N^2 K \qquad [19]$$

K is a constant (**Figure 6**) depending on the ratio of the diameter to the length of the coil. This calculation shows that a 1 meter long coil, with radius 1 meter and a single turn has an inductance of 2.075 microHenries, which is in good agreement with the Wheeler formula above. For very short coils interpolation of the factor K from the graph becomes difficult and a series formula is available, in Grover [2, pp. 143].

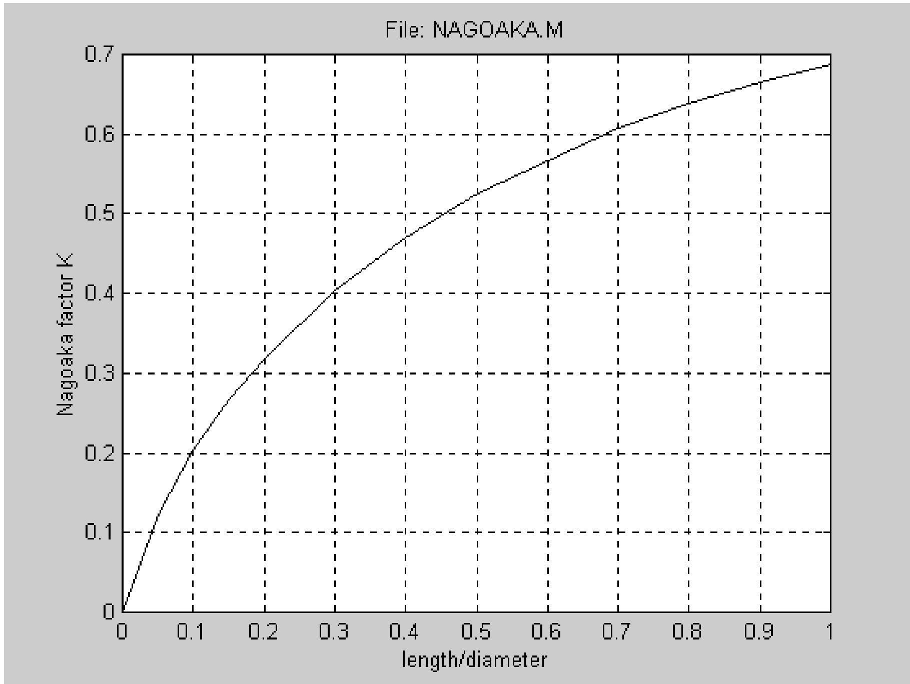


Figure 6. Correction factor for finite-length solenoid

4. Round Planar Spirals

Planar spiral coils have increasing application in miniature power electronics and in PC-board RF inductors. A number of methods are available for the calculation of the inductance of a round spiral coil. Using the Grover method [2, pp. 110] we find

$$L = \frac{25\mu_o}{\pi} aPN^2 \qquad [20]$$

where a is the mean coil diameter in meters and P is the factor depending on $c/2a$, as stated before. This equation is applicable if the inner and outer radii of the coil are not too different. For a circular coil with outer radius R_o and number of turns N, Schieber [16] calculates:

$$L = 1.748 \times 10^{-5} \mu_o \pi R_o N^2 \qquad [21]$$

where R_o is in meters and L is in Henries. This equation is suitable if windings are used over the entire area. Wheeler gives [5]:

$$L = 31.33 \mu_o N^2 \frac{a^2}{8a + 11c} \qquad [22]$$

where a is the coil mean radius, and c is the thickness of the winding. Wheeler states that the formula is correct to within 5% for coils with $c > 2a$. Other workers have reported errors using Wheeler's equation of < 20% [17]. Errors occur when there are few turns, or if the spacing between the turns is too great. For a spiral coil with outer radius R_o = 0.125" and inner radius $R_i = 0$, with N=5 calculation from the Grover method gives $L \approx$

55 nH, the Schieber method gives $L \approx 55$ nH and the Wheeler formula gives $L \approx 67$ nH. It appears that the Wheeler formula is more accurate.

5. Planar Square Coil

For the square coil, the effects of mutual coupling are not as simple to calculate as for the spiral case and the inductance is more difficult to calculate analytically. An empirical approximation for an *N*-turn square spiral (**Figure 7**) is given in [18]. It is reported in the same paper that a ratio of D/D_i of 5 optimizes the Q of the coil.

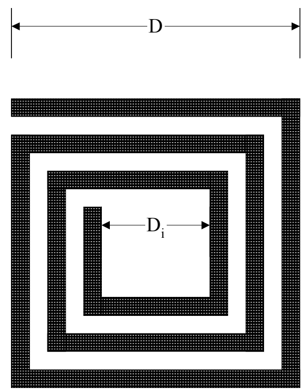


Figure 7. Planar Square Coil

In the case when the winding area is completely filled, or $D_i = 0$, the inductance is:

$$L \approx 8.5 \times 10^{-10} D N^{5/3} \qquad [23]$$

The exponent of 5/3 is thought to be due to end effects in the square coils. Note that this simplified expression doesn't take into account the trace width. More detailed inductance calculations for square planar spirals are given in references by Bryan [19], Greenhouse [20], Corkhill [21] and Saleh [22]. The Greenhouse reference is of special use as it compares various methods of calculation of planar spiral coils (both round and square) with experimental results.

V. CONCLUSIONS

Part II of this work has shown a variety of handbook and approximate inductance calculation results for different shaped conductors. A comprehensive set of references is listed if the reader wishes to delve into the topic in more detail. A bibliography covering inductance calculation techniques may be found at the author's business website [23] at:

http://members.aol.com/marctt/Technical/Inductance_References.htm

The author welcomes comments on this article and additions to the references at *marctt@aol.com.*

VI. REFERENCES

[1] James C. Maxwell, *Electricity and Magnetism*, vols. 1 and 2, reprinted in Dover Publications, 1954

[2] Frederick W. Grover, *Inductance Calculations: Working Formulas and Tables*, Dover Publications, Inc., New York, 1946

[3] U. S. Bureau of Standards, *Radio Instruments and Measurements*, Government Printing Office, 1924

[4] Harold A. Wheeler, "Formulas for the Skin Effect," *Proceedings of the I.R.E.*, September 1942, pp. 412-424

[5] Harold A. Wheeler, "Simple Inductance Formulas for Radio Coils," *Proceedings of the I.R.E.*, vol. 16, no. 10, October 1928

[6] Frederick E. Terman, *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943

[7] Herbert B. Dwight, *Electrical Coils and Conductors*, McGraw-Hill, 1945

[8] Herbert C. Roters, *Electromagnetic Devices*, John Wiley, 1961

[9] William R. Smythe, *Static and Dynamic Electricity*, 2d ed., McGraw-Hill, New York, 1950

[10] D. Bruce Montgomery, *Solenoid Magnet Design*, Wiley-Interscience, 1969

[11] Thomas H. Lee, *The Design of CMOS Radio Frequency Integrated Circuits*, Cambridge University Press, 1998

[12] Markus Zahn, *Electromagnetic Field Theory: A Problem Solving Approach*, John Wiley, 1979

[13] Marc T. Thompson, "High Temperature Superconducting Magnetic Suspension for Maglev," Ph.D. Thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, May 1997

[14] P. Murgatroyd, "The Brooks Inductor: A Study of Optimal Solenoid Cross-Sections," *IEE Proceedings*, vol. 133, part B, no. 5, September 1986, pp. 309-314

[15] V. J. Bashenoff, "Abbreviated Method for Calculating the Inductance of Irregular Plane Polygons of Round Wire," Proceedings of the I.R.E., vol. 15, 1927, pp. 1013-1039

[16] D. Schieber, "On the inductance of printed spiral coils," *Archiv fur Elektrotechnik*, 68 (1985) pp. 155-159

[17] D. Daly, S. Knight, M. Caulton, and R. Ekholdt, "Lumped Elements in Microwave Integrated Circuits," *IEEE Transactions on Microwave Theory and Techniques*, vol. MTT-15, no. 12, December 1967, pp. 713-721

[18] H. Dill, "Designing Inductors for Thin-Film Applications?" *Electronic Design*, Feb. 17, 1964, pp. 52-59

[19] H. E. Bryan, "Printed Inductors and Capacitors," *Tele-Tech and Electronic Industries*, vol. 14, no. 12, December 1955, pp. 68

[20] H. Greenhouse, "Design of Planar Rectangular Microelectronic Inductors," *IEEE Transactions on Parts, Hybrids, and Packaging*, vol. PHP-10, no. 2, June 1974, pp. 101-109

[21] J. Corkhill and D. Mullins, "The properties of thick film inductors," *Electronic Components*, vol. 10, no. 5, 1969, pp. 593

[22] N. Saleh, "Variable Microelectronic Inductors," *IEEE Transactions on Components, Hybrids, and Manufacturing Technology*, vol. CHMT-1, no. 1, March 1978, pp. 118-124
[23] Marc T. Thompson, business website *http://members.aol.com/marctt/Technical/Inductance_References.htm*

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