ALGEBRA — INTRODUCTION

Refresher on Notation

- Algebra $\rightarrow$ the *language* of mathematics.
- Use algebraic symbols or "pronumerals" $(x, y, z, \rho, \tau, \dots)$ to represent Real Numbers. You must realise that:

$$\boxed{\text{Rules for Algebra} \iff \text{Rules for Arithmetic}}$$

- Symbols allows us to write down formulae that describe various laws or relationships (e.g., mechanical, biological, economic), such as

$$A = \pi r^2 \text{ (Area of circle)}, \quad T = 2\pi\sqrt{\frac{\ell}{g}} \text{ (Pendulum)}$$

These are *general:* any feasible values may be substituted to obtain the answer for a given situation.

We later progress to the concept of a mathematical *function.* For example, $A = \pi r^2$ states that area A is a function of (i.e., depends on) radius, r. We might write this formula as

$$\begin{array}{llll}
A = & f(r) & = \pi r^2 & \\
& f(1) & = \pi \cdot 1^2 & = \pi \approx 3.142 \\
& f(3) & = \pi \cdot 3^2 & = 9\pi \\
& f(0) & = \pi \cdot 0^2 & = 0 \\
& f(a) & = \pi \cdot a^2 & = \pi a^2 \\
& f(2r) & = \pi \cdot (2r)^2 & = \pi \cdot 4 \cdot r^2 & \text{(Index Laws)} \\
& & = 4\pi r^2 & = 4f(r) & \text{(Meaning?)}
\end{array}$$

In algebra, we use both *Variables* and *Constants*.

Constant	*Variable*
Fixed value e.g., $2, \pi, 17.4, \ldots$	Value can vary over a certain domain
Usually start of alphabet e.g., $a, b, c, \alpha, \ldots$	Usually (not always) at end e.g., $x, y, t, \phi, \ldots$

Note: Letter/symbols/pronumerals represent Real Nos. They obey rules for $+$, $-$, $\times$, $\div$, and the Index Laws.

Multiplication: Two symbols (or a number and a symbol), written together $\Rightarrow$ multiplication:

$$
\begin{aligned}
xy &= x \cdot y \\
abc &= a \cdot b \cdot c \\
2x &= 2 \cdot x = x + x \\
5w &= 5 \cdot w = w + w + w + w + w
\end{aligned}
$$

Division: $$x \div y = \frac{x}{y} = x/y.$$

Expressions: A group of symbols and operators, written together, e.g.

$$2x + 3yz - w^2$$

is an *expression*. The components of an expression ($2x$, $3yz$, $-w^2$) are called *terms.*

Coefficients: In the term $2x$ we say that 2 is the *coefficient* of x.

Class Exercise.

In	$5y$,	the coefficient of	y	is	5
In	$12pq$,	the coefficient of	pq	is	
In	$4x^3$,	the coefficient of	x^3	is	
In	t^2,	the coefficient of	t^2	is	
In	$-7abc$,	the coefficient of	abc	is	
In	$-z$,	the coefficient of	z	is	

EQUATIONS

Equations say two expressions are *conditionally* true. Consider the equation

$$2x + 1 = 7.$$

- The *solution* (or solutions — only one here) is the number x that makes LHS = RHS.

- In other words, we want x so that twice this number plus one is 7. Clearly, $x = 3$ is *the* solution.

Identities are mathematical statements that are always true, regardless of the values represented by the pronumerals, e.g.

$$x + y = y + x \quad \text{or} \quad 3p = p + p + p.$$

There is NO solution to an identity. Other examples might include:

$$a^2 - b^2 = (a + b)(a - b)$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$\sin^2\theta + \cos^2\theta = 1$$

Grouping (Addition of like terms)

We know	$y + y + y$	$= 3 \times y = 3y$
and	$3x + 2x$	$= 5x$

The last example states: 3 apples plus 2 apples gives 5 apples; we *group* the like terms (multiples of x). However, the expression $3x + 2y$ cannot be simplified.

Example. $3p - 2p - 2p = -1 \times p = -p.$

Class Exercises.

$$2ab + 3ab =$$
$$3w - 2w =$$
$$a + b + 2a + b + 3a - 4b =$$
$$-9y + 3y =$$
$$2ab + 5ba =$$
$$x^2y - 2xy^2 + 3xy^2 + 2x^2y =$$

Multiplication, Division: These are illustrated by examples/exercises; attempt those that are incomplete.

$$(+a)(+b) = +a \times b = ab$$
$$(3p)(-2q) =$$
$$-(a)(-b) =$$
$$(-x)^2 = (-x) \cdot (-x) = x^2$$
$$\frac{6a}{3a} = \frac{\overset{2}{\cancel{6}} \times \cancel{a}}{\cancel{3} \times \cancel{a}} = 2 \qquad (6a \text{ is twice } 3a)$$
$$\frac{4ab}{-2b} = -\frac{}{} =$$

Index Laws: These have been given previously, and will not be repeated here. Most students find this work difficult, so considerable attention should be given to the Index Law problems on Tutorial Sheet 2. Some examples and exercises follow.

Examples. Simplify the following, expressing answers with positive indices.

1. $$\frac{a^6 \times a^3}{a^4} = \frac{a^{6+3}}{a^4} = \frac{a^9}{a^4} = a^{9-4} = a^5$$

2. $$\frac{x^{-3}}{x^2} = x^{-3-2} = x^{-5} = \frac{1}{x^5}$$

3. $$\frac{(4x)^{3/2}}{\sqrt{x}} = \frac{4^{3/2}\,x^{3/2}}{x^{1/2}} = (4^{1/2})^3\,x^{(3/2-1/2)} = 2^3 \cdot x^1 = 8x$$

Class Exercises.

$$(3x^{1/2})^2 =$$

$$\frac{10y^5}{5y^3} =$$

$$\frac{(4ab^2)^2}{8a^3b} =$$

DISTRIBUTIVE LAW

Arithmetic examples of the Distributive Law have been covered earlier. Thus

$$6 \times (5 + 4) = 6 \times 5 + 6 \times 4\,.$$

With numbers, we evaluate the bracket first on the LHS. If this is algebraic, e.g., $x + 2y$, use the RHS half of the Distributive Law to write as

$$6(x + 2y) = 6x + 12y\,.$$

In general

$$\boxed{a(b + c) = ab + ac}$$

This is usually called *Expanding a Bracket.* The reverse is *Factorisation*, writing a sum as the product of factors. The Distributive Law applies for differences too, and the multiplicative terms can be in any order:

$$p(q - r) = (q - r)p = pq - pr = qp - rp.$$

<u>Examples.</u> Expand the following.

1. $3(m - 5) \quad = 3 \times m - 3 \times 5 = 3m - 15$
2. $a(a + 4b) \quad =$
3. $-2(x - 2y) \quad = (-2) \times x - (-2) \times 2y =$
4. $4(2a - b) - 3(a - 2b) \quad =$

Note: take care with *signs* – practise Nos. 3 and 4.

Factorisation

Factorisation is the reverse process of Expansion. Given the sum of two (or more) terms, factor out the HCF. With $6a - 8ab$ to factorise, first reduce each *additive* term to its basic factors:

$$6a = 2 \times 3 \times a \qquad \text{and} \qquad 8ab = 2^3 \times a \times b$$

Using the previous HCF rule, $\text{HCF} = 2 \times a = 2a$. Thus

$$6a - 8ab = 2a\left(\frac{6a}{2a} - \frac{8ab}{2a}\right) = 2a(3 - 4b)$$

The intermediate steps are usually omitted. Factorisation can *always* be verified by expanding the RHS — check that the original LHS is recovered. You must ensure the *Highest* CF, not just any common factor is used, or else the factorisation will be incomplete.

Examples. Factorise the following *completely*.

1. $2x + 6y \quad = 2(x + 3y) \quad$ (HCF of $2x$ and $6y$ is 2)
2. $3ab - 9ab^2 \quad = 3ab(1 - 3b) \quad$ (HCF is $3ab$)

If unsure, check your results by expanding

3. $x^2 + 5x \quad =$
4. $6y^3 - 18y^2 \quad =$
5. $-2x + 6 \quad = \; -2(\qquad)$

   or $\quad = \; 2(\qquad)$

We can sometimes use factorisation to simplify algebraic fractions, e.g. $\frac{9x+18}{3x+6}$. As an exercise, substitute some values for x. Try $x = 2$. Now, with $x = -5$:

$$\frac{9x + 18}{3x + 6} \quad (\text{with } x = -5) \quad \Rightarrow \quad \frac{-45 + 18}{-15 + 6} = \frac{-27}{-9} = 3,$$

the same answer as we get by substituting $x = 2$. Using factorisation, we see that

$$\frac{9x+18}{3x+6} = \frac{\overset{3}{\cancel{9}}\,(x \cancel{+2})}{\cancel{3}(x \cancel{+2})} = 3 \quad [\textbf{Except} \text{ if } x = -2.\ \text{Why?}]$$

BINOMIAL EXPANSION ("FOIL")

We now extend the Distributive Law to expand the product of two binomials. A *binomial* contains *two additive* terms, e.g. $(x+1)$, $(a-2b)$, $(yz-5w^2)$, etc. Consider the expansion of $(x+2)(x+4)$.

We can write the second bracket, $(x+4)$, as the term B, so that

$$\begin{aligned}(x+2)(x+4) &= (x+2)B \\ &= B(x+2) \\ &= Bx + 2B = xB + 2B \\ &= x(x+4) + 2(x+4)\end{aligned}$$

We *don't* want to take all these steps in general, but simply need to say that the expansion is $[x \times \text{Bracket} + 2 \times \text{Bracket}]$.

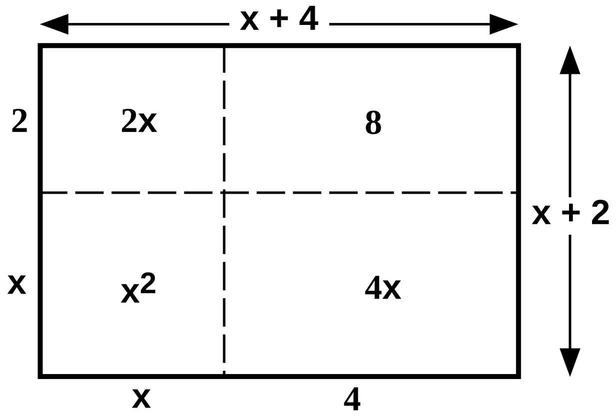
$$\begin{aligned}\therefore\ \ (x+2)(x+4) &= x(x+4) + 2(x+4) \\ &= x^2 + \underbrace{4x + 2x} + 8 \\ &= x^2 + 6x + 8 \quad [\text{Grouping}\ \ (4x+2x)]\end{aligned}$$

The product of 2 terms (additive or subtractive) times 2 more gives $2 \times 2 = 4$ terms. By grouping 2 terms the final result contains 3 terms for <u>this</u> example. A useful way to remember the method is:

FOIL: *First* *Outer* *Inner* *Last*

$$x + 2)(x + 4) = \underset{F}{x^2} + \underset{O}{4x} + \underset{I}{2x} + \underset{L}{8}$$

Geometry



In the above, the total area is $(x + 2)(x + 4)$. This is comprised of 4 sub-sections, with individual areas of x^2, $4x$, $2x$ and 8.

<u>Example.</u> Be careful with signs.

$$\begin{aligned}(a - 4)(2a - 3) &= a(2a - 3) - 4(2a - 3)\\ &= 2a^2 - 3a - 8a - 4(-3)\\ &= 2a^2 - 11a + 12\end{aligned}$$

Note the $+12$ in the answer.

<u>Class Exercises.</u>

1. $(x - 3)(x + 2) \quad =$
$\qquad\qquad\qquad\quad =$

2. $(x + 4)^2 \quad = (x + 4)(x + 4)$
$\qquad\qquad\quad =$

Note: There are 3 special binomials. These arise so often that their expansions (which can all be worked out by FOIL) should definitely be committed to memory.

- $(a+b)^2 \quad = \quad a^2+2ab+b^2$ (Sum squared)
- $(a-b)^2 \quad = \quad a^2-2ab+b^2$ (Difference squared)
- $(a+b)(a-b) \quad = \quad a^2-b^2$ (Difference of squares)

Example.
$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ba-b(-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

Note the $\mathbf{2}ab$ term. You must know $(a+b)^2 \neq a^2+b^2$ and $(a-b)^2 \neq a^2-b^2$.

We can use the above formulae to work out some more slightly challenging arithmetic problems.

Example.
$$\begin{aligned}63^2 &= (60+3)^2\\ &=\\ &=\\ &=\end{aligned}$$

Evaluate **Trinomial** and similar products also by using the Distributive Law. Each (additive) term in the first bracket multiplies each in the second. Then add all together; e.g., a binomial (2 terms) $\times$ a trinomial (3 terms) $\rightarrow 2\times 3=6$ terms – you may be able to be group some together. Trinomial $\times$ trinomial $\rightarrow$ 9 terms initially, e.g.

$$\begin{aligned}(a-2b-3)(3a+b-2) &= a(3a+b-2)-2b(3a+b-2)-3(3a+b-2)\\ &=\\ &=\end{aligned}$$

Binomial Theorem

We have used the Distributive Law to expand products of binomials. A typical product is

$$(x+3)(2x-7) = 2x^2 - 7x + 6x - 21 = 2x^2 - x - 21.$$

We also studied products of trinomials — 3 (additive) terms. Three special binomial products were noted, including squares of binomials, $(a+b)^2$ and $(a-b)^2$:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

The Binomial Theorem gives a method for calculating higher powers of binomials, e.g., $(3-2x)^5$. It is a formula for $(a+b)^n$ (n is a +ve integer), without explicitly working out the expansion. We first experiment with results for $(a+b)^n$ (choose $n = 0, 1, 2, 3, \ldots$):

$\underline{n=0}$: $\quad (a+b)^0 = 1.$

$\underline{n=1}$: $\quad (a+b)^1 = a+b.$

$\underline{n=2}$: $\quad (a+b)^2 = a^2 + 2ab + b^2.$ (prev. lecture)

$\underline{n=3}$:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a^3 + \underline{2a^2b} + \underline{\underline{ab^2}} + \underline{a^2b} + \underline{\underline{2ab^2}} + b^3.\end{aligned}$$

Collect like terms:

$$= a^3 + 3a^2b + 3ab^2 + b^3.$$

$\underline{n=4}$:

$$\begin{aligned}(a+b)^4 &= (a+b)^2(a+b)^2 \\ &= (a^2+2ab+b^2)(a^2+2ab+b^2) \\ &= \\ &= \end{aligned}$$

Verify that the result is

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Try to show (the Binomial Theorem allows us to avoid such tedious calculations) that:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The above are now rewritten, to help identify the various patterns in the expansions:

$$\begin{aligned}
(a+b)^0 &= 1 \\
&= \mathbf{1}a^0b^0 \\
(a+b)^1 &= a+b \\
&= \mathbf{1}a^1b^0 + \mathbf{1}a^0b^1 \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
&= \mathbf{1}a^2b^0 + \mathbf{2}a^1b^1 + \mathbf{1}a^0b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
&= \mathbf{1}a^3b^0 + \mathbf{3}a^2b^1 + \mathbf{3}a^1b^2 + \mathbf{1}a^0b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
&= \mathbf{1}a^4b^0 + \mathbf{4}a^3b^1 + \mathbf{6}a^2b^2 + \mathbf{4}a^1b^3 + \mathbf{1}a^0b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
&= \mathbf{1}a^5b^0 + \mathbf{5}a^4b^1 + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}a^1b^4 + \mathbf{1}a^0b^5
\end{aligned}$$

<u>Evident patterns</u>:

- $(n+1)$ terms: $(a+b)^2$ has 3 terms, $(a+b)^5$ has 6, etc.
- Powers of a: $n \to 0$; powers of b: $0 \to n$.
- The coefficients of each term show a special pattern, given by the numbers in *Pascal's Triangle.*
- * Sum adjacent terms in a row; compare with term immediately below — *generates* Pascal's triangle.
- Sum the terms in each row.

Pascal's Triangle

0								1							
1							1		1						
2						1		2		1					
3					1		3		3		1				
4				1		4		6		4		1			
5			1		5		10		10		5		1		
6		1												1	
7	1														1

Example. Expand $(x+2)^3$ — use the $(a+b)^3$ formula.

Here, set $a = x$ and $b = 2$ and use the [1 3 3 1] line of Pascal's triangle.

$$\begin{aligned}(x+2)^3 &= (x)^3 + 3\cdot(x)^2\cdot(2) + 3\cdot(x)\cdot(2)^2 + (2)^3 \\ &= x^3 + 6x^2 + 3x\cdot(4) + 8 \\ &= x^3 + 6x^2 + 12x + 8.\end{aligned}$$

Class Exercises. Expand using the Binomial Theorem.

1. $(1+x)^4$. [Here, $a = 1$ and $b = x$.]

$$\begin{aligned} &= (1)^4 + 4\cdot(1)^3\cdot(x) + \qquad + \qquad + \\ &= \\ &= \end{aligned}$$

2. $(2p-3)^4$. [$a = 2p$, $b = -3$, as $2p - 3 = 2p + (-3)$.]

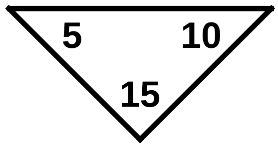
$$\begin{aligned} &= \\ &= \\ &= \end{aligned}$$

<u>Binomial Coefficients</u> are the nos. in P's Δle: write ${}^nC_r$ or $\binom{n}{r}$; n is the power, r ranges from 0 to n. ${}^nC_r$ is the no. of *Combinations* in choosing r objects from n — the no. of ways r can be chosen at random from n: (i) without replacement, and (ii) without regard to order. Generate Pascal's triangle, by using the relationship:

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}, \quad \text{or} \quad \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

Show digramatically as:

e.g. (*n = 5, r = 1*)



Choose 2 objects from 5 $\Rightarrow {}^5C_2$ combinations: Pascal's triangle (row 5) $\rightarrow$ 10 ways. Now check this directly.

Given 5 objects (A, B, C, D, E), there are 5 ways to choose the 1^{st}. This is *not* replaced, so (for *each* of the 5 ways), there are 4 ways to choose the 2^{nd}; i.e., a total of $5 \times 4 = 20$ ways to choose 2 from 5 in an <u>ordered</u> manner (the no. of *Permutations*). But, order is not important with Combinations (choosing 'AB' = choosing 'BA'). Therefore, we divide 20 by 2 to get ${}^5C_2 = 10$.

Before proceeding, we define *Factorial Numbers,* written as $n!$. These are the product of all positive integers up to and including n, e.g.,

$$1! = 1$$
$$2! = 2 \cdot 1 = 2$$
$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120, \quad \text{etc.}$$

Any factorial can be found from the value of the previous factorial, e.g., $5! = 5 \cdot 4!, 6! = 6 \cdot 5! = 720$. In general,

$$n! = n\,(n-1)! \qquad \text{or} \qquad (n-1)! = \frac{n!}{n}\,.$$

This allows us to extend factorials to include 0!. Clearly, if 0! is *consistent* with the rest, then

$$0! = \frac{1!}{1} = \frac{1}{1} = 1.$$

Returning to ${}^5C_2$, we see this can be written as

$${}^5C_2 = \frac{5 \cdot 4}{2} = \frac{5 \cdot 4 \cdot (3 \cdot 2 \cdot 1)}{(2 \cdot 1)(3 \cdot 2 \cdot 1)} = \frac{5!}{2!\,3!}\,.$$

To find ${}^nC_r$, the top line is $n!$, the bottom is $r! \times (n-r)!$. Thus ${}^5C_2 = {}^5C_3$, since $2+3=5$. This relationship (${}^nC_r = {}^nC_{n-r}$) is seen in the *symmetry* of Pascal's triangle about its centreline. This is logical since by choosing 2 from 5, we automatically choose 3 from 5 (those left behind).

Class Exercises. Calculate ${}^4C_1$ and ${}^6C_4$.

${}^4C_1 = \underline{\qquad\qquad} =$ ${}^6C_4 = \underline{\qquad\qquad} =$

Calculators. Most calculators include the ${}^nC_r$ and ${}^nP_r$ (permutation) functions. So, there are 3 ways to find binomial coefficients: (i) Pascal's triangle, (ii) factorial formula, or (iii) calculator. The second method, i.e.,

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}\,,$$

can be tedious for large numbers as most of them cancel. However, there is a shortcut, as seen from the example of ${}^{20}C_3$. *(Note: 3 numbers on top, 3 on the bottom.)*

$${}^{20}C_3 = {}^{20}C_{17} = \frac{20!}{3!\,17!} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140.$$

Check the result on your calculator. it is clearly impractical to construct Pascal's triangle here. (You *must* also get a positive integer for your answer, not a fraction.)

Binomial Theorem: To give a compact formula for the theorem, we first introduce the *Summation Convention*. This is used to write SUMS concisely, e.g., the sum of squares of the first 100 positive integers as:

$$1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2 = \sum_{k=1}^{100} k^2 = 338\,350.$$

Here, Σ (sigma) stands for "sum". We add all values of k^2, from $k = 1$ to $k = 100$: let $k = 1$ and find $k^2 = 1^2 = 1$, then let $k = 2$ and *add* $k^2 = 2^2 = 4$ to get 5, then add 9 ($= 3^2$), etc., up to 100^2. Note that

$$\sum_{k=1}^{100} k^2 \;=\; \sum_{r=1}^{100} r^2 \;=\; \sum_{j=1}^{100} j^2$$

so that each "summation index", (k, r, j, or whatever) is only a *dummy index* and doesn't affect the answer.

<u>Class Exercise.</u> Write the sum of the first 50 positive integers using Σ notation. Can you evaluate the sum? If so, try $\sum_{i=1}^{1000} i$.

The **Binomial Theorem** can now be expressed as:

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^r}$$

$$a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n.$$

Factorising Quadratics

A quadratic is an expression such as $x^2 - 5x + 6$. It contains (additive) terms, where the variable x has a highest power of 2 (x^2), plus powers of 1 ($x^1 = x$), and zero ($x^0 = 1$). We could write $x^2 - 5x + 6$ as

$$(+1) \cdot x^2 + (-5) \cdot x + (+6) \cdot x^0 .$$

The coefficients of this quadratic are the *ordered set* $\{1, -5, 6\}$. A *General Quadratic* with coefficients a, b and c has the form

$$\boxed{ax^2 + bx + c} \qquad (a \neq 0)$$

Note that if $a = 0$ we do not have a quadratic, but a *linear* expression, where the highest power of x is 1.

Examples.

$$\begin{array}{cl} x^2 + 3x - 4 & (a = \ \ 1, b = \ \ 3, c = -4) \\ 3x^2 + x - 7 & (a = \ \ 3, b = \ \ 1, c = -7) \\ -4x^2 - x - 16 & (a = \qquad, b = \qquad, c = \qquad) \\ x^2 - 4 & (a = \qquad, b = \qquad, c = \qquad) \end{array}$$

As above, factorising is the reverse of multiplication. Given two linear factors, we can expand:

$$(x + 1)(x + 4) =$$

To factorise, we do the reverse process, i.e., given a quadratic, write it as the product of two linear factors.

i.e., $x^2 + 5x + 6 = (\quad \mathbf{?} \quad)(\quad \mathbf{?} \quad)$

Terms in x and constants ONLY.

In expanding by FOIL, the x^2 term can come *only* from the product of x times x. Therefore

$$x^2 + 5x + 6 = (x + \mathbf{?})(x + \mathbf{?}) \quad \text{or} \quad (x - \mathbf{?})(x - \mathbf{?}).$$

Note: the signs in the factors must be *like*, so that the product of the 2 numbers is $+6$. Also, their sum must equal 5. The two numbers can only be $+3$ and $+2$.

$$\therefore \quad x^2 + 5x + 6 \quad \text{factors as} \quad (x+3)(x+2).$$

X–Method — Cover the example x^2+6x+8 in lectures.

(i) $x \times x \longrightarrow x^2$.

(ii) $+8 \Leftrightarrow$ *Same Signs:*$(+)(+) \to (+)$ or $(-)(-) \to (+)$.

(iii) $+6x \Leftrightarrow$ *Both Numbers Positive* (sum $= +6$); if the term is $-6x$, both numbers must be Negative.

(iv) Product $= 8$; sum of numbers (same sign) $= +6$.

Class Exercises. Factorise the following:

1. $x^2 - 7x + 10 = (x - \quad)(x - \quad)$.
2. $x^2 - 9x + 18 = (\qquad)(\qquad)$.
3. $x^2 + 3x - 10 = (\qquad)(\qquad)$.
4. $x^2 \;\; - x \;\; - 2 = (\qquad)(\qquad)$.

Discriminant Test: tells if a quadratic will reduce to 2 linear factors (comes from the *Quadratic Formula*), e.g.

$$x^2 + 2x - 2.$$

(We want Product $= 2$; Difference $= 2$ — Not Possible). Also, it may be hard to tell if a quadratic is factorable:

$$x^2 - 6x - 135 = (x - 15)(x + 9).$$

- A quadratic, $ax^2 + bx + c$, can *always* be factored into two "rational" linear factors, *provided:*

$$\boxed{\sqrt{b^2 - 4ac} \qquad \text{is an integer.}}$$

Examples/Exercises. Use the Discriminant test for the following quadratics, and factorise where possible.

1. $x^2 - 2x - 24$ $(a = 1, b = -2, c = -24)$

$$\Delta = \sqrt{b^2 - 4ac} = \sqrt{(-2)^2 - 4(1)(-24)}$$
$$= \sqrt{4 + 96} = \sqrt{100} = 10.$$

Always use brackets to find Δ

Here $x^2-2x-24 = (x-6)(x+4) = (x-6)(x-(-4))$.

Note: the difference between 6 *and* -4 *is* 10.

2. $x^2 + x + 1$ $(a = 1, b = 1, c = 1)$

$$\sqrt{b^2 - 4ac} = \sqrt{(1)^2 - 4(1)(1)} = \sqrt{-3}$$ **(Impossible!)**

Therefore, $x^2 + x + 1$ cannot be factorised.

3. $x^2 - 8x + 15$ $(a = \quad , b = \quad , c = \quad)$

$\sqrt{b^2 - 4ac} =$

$\therefore \quad x^2 - 8x + 15 = (x \qquad)(x \qquad)$.

4. $x^2 - 9x$ $(a = \quad , b = \quad , c = \quad)$

$\sqrt{b^2 - 4ac} =$

$\therefore \quad x^2 - 9x =$

5. $x^2 - 16 =$ $(a = \quad , b = \quad , c = \quad)$

Special Case:– Coefficient of x^2 is not unity $\quad (a \neq 1)$.

e.g. $\quad 2x^2 + 7x + 3 \qquad (a = 2, b = 7, c = 3)$

Previous work should allow us to assume that

$$2x^2 + 7x + 3 = (\underline{2x+}\ \textbf{?}\ )(\underline{x+}\ \textbf{?}\ )$$

$$\sqrt{b^2 - 4ac} = \sqrt{(7)^2 - 4(2)(3)} = \sqrt{49 - 24} = \sqrt{25} = 5.$$

The $\sqrt{b^2 - 4ac}$ test also works here, but we need to *find* the two "rational" factors (*all* terms have <u>integer</u> coefficients). From the test, we should have either

$$\left.\begin{aligned} & 2x^2 + 7x + 3 = (2x + 3)(x + 1), \\ \textbf{or} \quad & 2x^2 + 7x + 3 = (2x + 1)(x + 3). \end{aligned}\right\}$$

Test which one is correct by using FOIL.

<u>Examples/Exercises.</u>

1. $\quad 3x^2 - 10x + 3 \qquad (a = \quad , b = \quad , c = \quad)$

$\sqrt{b^2 - 4ac} =$

$\therefore \quad 3x^2 - 10x + 3 = (3x \qquad)(x \qquad).$

2. $\quad 6x^2 - 5x - 6 \qquad (a = \quad , b = \quad , c = \quad)$

(Note: considerable choice exists in this case for the coefficients, even if the test is successful.)