

THE UNIVERSITY OF AKRON
Department of Theoretical and Applied Mathematics

LESSON 2:
THE TRIGONOMETRIC FUNCTIONS

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Last Revision Date: August 17, 2001

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1. Introduction

There are six trigonometric functions: **sine**, **cosine**, **tangent**, **cotangent**, **secant**, and **cosecant**; abbreviated as sin, cos, tan, cot, sec, and csc respectively. These are functions of a single real variable that is normally an angle measurement given in terms of radians or degrees. Consequently, values such as $\sin 2.7$ rad or $\tan 33^\circ$ (read: the sine of 2.7 radians or the tangent of 33 degrees) often appear in trigonometric expressions. In the first case, the radian identifier (rad) is frequently suppressed for simplicity and $\sin 2.7$ rad is shortened to $\sin 2.7$. When variables such as t or α (α is the Greek letter alpha) denote angles, measurement identifiers are usually omitted. Consequently, the reader will encounter expressions such as $\sin t$ and $\tan \alpha$. In such cases the context must make the choice of measurement clear. In this tutorial we will normally use Greek letters to denote angles measured in degrees while most other variables generally denote radian measurement.

The [next section](#) of this lesson introduces the trigonometric functions using their *circular* definitions. Specifically, the functions are defined by coordinates of points on a particular circle. This approach requires that the angles constructed by points on this circle be given in radian measure. [Section 3](#) and [Section 4](#) demonstrate strategies for computing the numerical values of the trigonometric functions at certain *special angles* by using various geometric properties of the circle. The [final section](#) of this lesson examines the **periodic** or cyclic nature of the trigonometric functions.

It should be noted that right triangles provide an alternate means for defining

the trigonometric functions. In this setting the angles are normally measured in degrees and the length of the sides of right triangles are used to determine the values of the trigonometric functions. This approach and its connection with the circular definitions is presented in a subsequent lesson. (See [Lesson 3](#).)

2. Definition of the Trigonometric Functions

As indicated in [Figure 2.1](#) let P be the point on the unit circle with coordinates (x, y) that determines the angle in standard position¹ of measure t rad. Then the values of all six trigonometric functions at t are defined in terms of the numbers x and y . For example, the sine function is defined to have value y at the angle of measure t radians. That is,

$$\sin t = y.$$

Example 1 *Since the coordinates of the point P on the unit circle determined by the angle 0 rad are $(1, 0)$ we see that $\sin 0 = 0$.*

The cosine function is defined by

$$\cos t = x.$$

Example 2 *Again, since $P(1, 0)$ is the point on the unit circle described by the angle 0 rad we have $\cos 0 = 1$.*

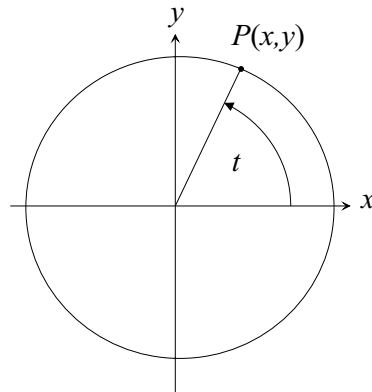


Figure 2.1: Example of an angle determined by a point on the unit circle.

¹An angle is in standard position if its initial side lies on the x -axis and has its vertex at the origin.

The definitions of all six trigonometric functions are given in [Table 2.1](#) below. The student should memorize these definitions. A few *elementary identities* or relations

$\sin t = y$	$\cos t = x$	$\tan t = \frac{y}{x}$	$\cot t = \frac{x}{y}$	$\sec t = \frac{1}{x}$	$\csc t = \frac{1}{y}$
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Table 2.1: Definition of the trigonometric functions.

easily follow from these definitions. For example,

$$\begin{aligned}\csc t &= \frac{1}{y} = \frac{1}{\sin t} \implies \sin t = \frac{1}{\csc t} \\ \sec t &= \frac{1}{x} = \frac{1}{\cos t} \implies \cos t = \frac{1}{\sec t} \\ \tan t &= \frac{y}{x} = \frac{1}{\cot t} = \frac{\sin t}{\cos t} \quad \text{and} \\ \cot t &= \frac{x}{y} = \frac{1}{\tan t} = \frac{\cos t}{\sin t}.\end{aligned}$$

Identities such as these have an important role in the study and use of trigonometry. [Lesson 6](#) is devoted to their development.

3. Some values of the trigonometric functions

It is relatively easy to calculate the values of the trigonometric functions for particular angles. For example, we saw in [Example 1](#) that for $t = 0$ the coordinates of P in [Figure 2.1](#) are $(1, 0)$ so $x = 1$ and $y = 0$. Consequently,

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \tan 0 = 0, \quad \text{and} \quad \csc 0 = 1.$$

The cotangent and secant functions are *undefined* when $t = 0$ because division by zero is undefined.

At $t = \pi/2$ the point P has coordinates $(0, 1)$, so

$$\sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0, \quad \cot \frac{\pi}{2} = 0, \quad \text{and} \quad \csc \frac{\pi}{2} = 1.$$

The tangent and secant functions are undefined at $\pi/2$.

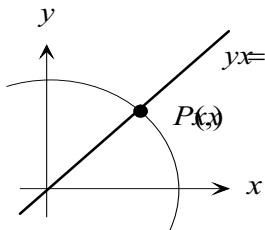
Remember: $\sin t = y$, $\cos t = x$, $\tan t = \frac{y}{x}$, $\cot t = \frac{x}{y}$, $\sec t = \frac{1}{x}$, $\csc t = \frac{1}{y}$.

We have seen that it is easy to find the values of the trigonometric functions at $t = 0$ and $\pi/2$. There are other important angles for which the values of these functions can be directly calculated. Some techniques for computing these values are illustrated in the following examples. These *special values*² are summarized in [Table 2.2](#) on page 13. They should be memorized and the techniques for computing them should be mastered. Memorizing these values and mastering the geometric principles employed to compute them will assist the reader in developing an understanding of the basic principles of trigonometry, and in acquiring the necessary skills for solving more complicated trigonometric problems.

²The values of the trigonometric functions for arbitrary angles are not easily computed and the introduction of the hand held calculator and other computing devices has greatly simplified this task. *When using a computing device the user must remember to set it to degree or radian mode depending on the dimension being used to measure angles.*

Example 3 Determine the values of the trigonometric functions at $t = \pi/4$.

Solution: As suggested by the figure below, at $t = \pi/4$ (45°) the two coordinates x and y of the point $P(x, y)$ on the unit circle must be equal since P lies on the line $y = x$.



Replacing y by x in the equation of the unit circle ($x^2 + y^2 = 1$) yields $x^2 + x^2 = 1$, or $2x^2 = 1$. Then,

$$x^2 = 1/2 \quad \text{or} \quad x = \pm(1/\sqrt{2}).$$

The point P is in the first quadrant of the plane suggesting that x is positive so

$$x = \frac{1}{\sqrt{2}}.$$

Since $y = x$ we also have³

$$y = \frac{1}{\sqrt{2}}.$$

Appealing directly to the definitions of the trigonometric functions we have

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \tan \frac{\pi}{4} = \cot \frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1, \quad \text{and} \quad \sec \frac{\pi}{4} = \csc \frac{\pi}{4} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

³Note that since $\frac{\sqrt{2}}{\sqrt{2}} = 1$ we have

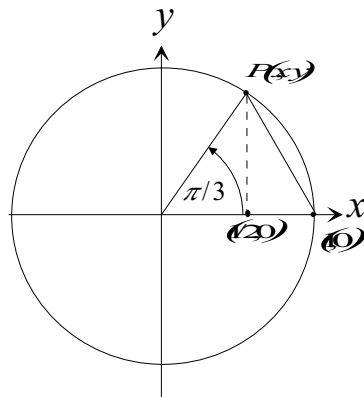
$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Historically, students were required to make conversions such as this. This process, referred to as *rationalizing the denominator*, often simplified numerical computations. Indeed, observe that without a calculator it is much more difficult to perform the division $\frac{1}{\sqrt{2}}$ than the division $\frac{\sqrt{2}}{2}$. Computing machines all but eliminated complicated hand calculations and the process of rationalizing the denominator no longer holds the prestigious position it once did in arithmetic computations. However, this process is often used as a procedure for simplifying algebraic expressions and remains a valuable tool for practitioners.

Example 4 Determine the values of the trigonometric functions at $t = \pi/3$.

Solution: Examine the larger triangle with vertices P , $(1, 0)$, and the origin in the figure to the right. The two radii of the unit circle forming the angle of measure $\pi/3$ rad each have length one so the triangle is isosceles. Since the remaining two angles must be equal and have sum⁴ $\pi - \frac{\pi}{3} = 2\pi/3$, they must also have measure $\pi/3$ rad. This means that the larger triangle is equilateral so each side has length one. Note the two right triangles forming the equilateral triangle with common side indicated by the dashed line. These triangles have base length $1/2$ because these two lengths must be equal and sum to one. That is the x -coordinate of the point P that determines the angle $\pi/3$ is $1/2$.

Appealing to the equation of the unit circle it follows that $(\frac{1}{2})^2 + y^2 = 1$ so $y = \sqrt{3}/2$. We have used positive roots since P is in the first quadrant. Hence,



$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos \frac{\pi}{3} = \frac{1}{2} \quad \tan \frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \quad \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}} \quad \sec \frac{\pi}{3} = 2 \quad \csc \frac{\pi}{3} = \frac{2}{\sqrt{3}}.$$

⁴The sum of the three angles of a triangle must be π rad or 180° .

Example 5 Determine the values of the trigonometric functions at $t = \pi/6$.

Solution: An argument similar to that used in the previous example ⁵suggests that the point P on the unit circle that determines the angle of $\pi/6$ rad has coordinates $(\sqrt{3}/2, 1/2)$. Consequently, the values of the trigonometric functions at $\pi/6$ rad are

$$\begin{aligned}\sin \frac{\pi}{6} &= \frac{1}{2} & \cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\ \tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}} & \cot \frac{\pi}{6} &= \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \\ \sec \frac{\pi}{6} &= \frac{2}{\sqrt{3}} & \csc \frac{\pi}{6} &= 2.\end{aligned}$$

Remember: $\sin t = y$, $\cos t = x$, $\tan t = \frac{y}{x}$, $\cot t = \frac{x}{y}$, $\sec t = \frac{1}{x}$, $\csc t = \frac{1}{y}$.

⁵For a different approach see [Exercise 4](#).

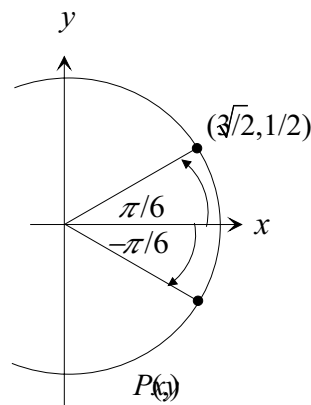
A summary of the values of the trigonometric functions at various angles, including those discussed thus far, is given in [Table 2.2](#) below. A dash (–) indicates that the function is undefined at the given angle.

Function Radians	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
0	0	1	0	–	1	–
$\pi/6$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\pi/4$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\pi/2$	1	0	–	0	–	1

Table 2.2: Values of the trigonometric functions at special angles.

Example 6 Determine the values of the trigonometric functions at $t = -\pi/6$.

Solution: Let P denote the point on the unit circle that determines the angle $-\pi/6$ rad. Consider the figure to the right and note the symmetric relationship between the coordinates of the two points determining the angles $\pi/6$ rad and $-\pi/6$ rad. Armed with the results of [Example 5](#), this suggests that the coordinates of P are $(\sqrt{3}/2, -1/2)$. Consequently,



$$\begin{aligned} \sin\left(-\frac{\pi}{6}\right) &= -\frac{1}{2} & \cos\left(-\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} \\ \tan\left(-\frac{\pi}{6}\right) &= -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}} & \cot\left(-\frac{\pi}{6}\right) &= -\sqrt{3} \\ \sec\left(-\frac{\pi}{6}\right) &= \frac{2}{\sqrt{3}} & \csc\left(-\frac{\pi}{6}\right) &= -2. \end{aligned}$$

Example 7 Find $\sin 30^\circ$ and $\tan(-30^\circ)$.

Solution: A 30° angle has radian measure $\pi/6$ so $\sin 30^\circ = \sin \frac{\pi}{6} = 1/2$. Likewise, $\tan(-30^\circ) = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$.

4. Trigonometric functions at related angles

Example 6 demonstrated that using the symmetry of the unit circle the values of the trigonometric functions at $\pi/6$ rad could be used to compute the values of these functions at $(-\pi/6)$ rad. This section is devoted to exploiting that and other symmetry properties of the unit circle. **Figure 2.2** suggests that the coordinates of points symmetrically located with respect to the x -axis are related. That is, reflecting the point (x, y) about the x -axis identifies the point $(x, -y)$ on the circle. Consequently, if the angle of measure t rad determines the point (x, y) on the unit circle, then this symmetry suggests that the angle of measure $(-t)$ rad identifies the point $(x, -y)$ on the circle. Hence,

$$\begin{aligned}\sin(-t) &= -y = -\sin t \quad \text{and} \\ \cos(-t) &= x = \cos t.\end{aligned}$$

A moment's reflection will convince the reader that these relationships hold regardless of the magnitude or direction of the angle t .

Similar identities hold for the remaining trigonometric functions. These can be established using the geometric symmetry of the unit circle. They can also be derived

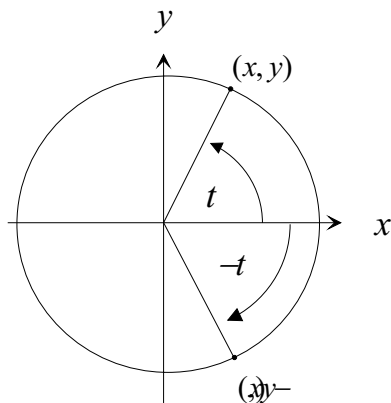


Figure 2.2: Symmetry properties of the angles t rad and $(-t)$ rad.

by appealing to the identities presented in [section 2](#). For example, using the identities above for the sine and cosine function and the identity

$$\tan t = \frac{\sin t}{\cos t}$$

we have

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\frac{\sin t}{\cos t} = -\tan t.$$

Example 8 *Using the information provided in [Table 2.2](#) we have the following values*

Function Radians	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
0	0	1	0	—	1	—
$-\pi/6$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2
$-\pi/4$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$-\pi/3$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
$-\pi/2$	-1	0	—	0	—	-1

The symmetry of the unit circle illustrated in [Figure 2.3](#) indicates that the angle of measure $(\pi - t)$ rad determines the point $(-x, y)$ on the unit circle if the angle of measure t rad determines (x, y) . Hence,

$$\begin{aligned}\sin(\pi - t) &= y = \sin t \quad \text{and} \\ \cos(\pi - t) &= -x = -\cos t.\end{aligned}$$

Once again, similar identities for the remaining trigonometric functions can be established using these symmetry properties or by direct calculation. Both strategies are illustrated in the next example.

Example 9 Use the symmetry properties of the unit circle and the methods of direct calculation to determine $\sec(\pi - t)$.

Solution: Here,

$$\sec(\pi - t) = \frac{1}{-x} = -\frac{1}{x} = -\sec t$$

$$\sec(\pi - t) = \frac{1}{\cos(\pi - t)} = \frac{1}{-\cos t} = -\frac{1}{\cos t} = -\sec t. \quad (\text{Direct Calculation})$$

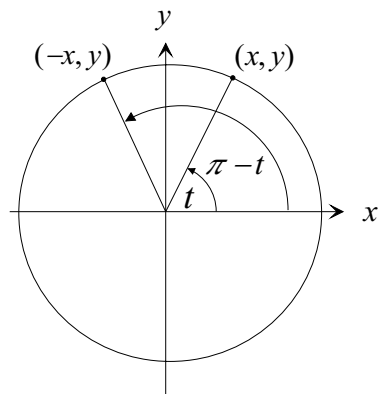


Figure 2.3: Symmetry properties of the angles t rad and $(\pi - t)$ rad.

(Symmetry)

Example 10 Using the symmetry of the circle and the information in [Table 2.2](#) determine the values of the sine, cosine, and tangent functions at a) $\frac{3\pi}{4}$ rad and b) $\frac{5\pi}{6}$ rad.

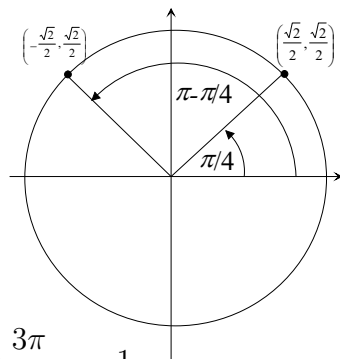
Solution:

a) Observe that $\frac{3\pi}{4} = \pi - \pi/4$. [Table 2.2](#) indicates that the angle $\frac{\pi}{4}$ rad corresponds to the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ on the unit circle, so it follows from the preceding discussion that $(\pi - \frac{3\pi}{4})$ rad determines the point $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. (The figure to the right may be helpful.)
Consequently,

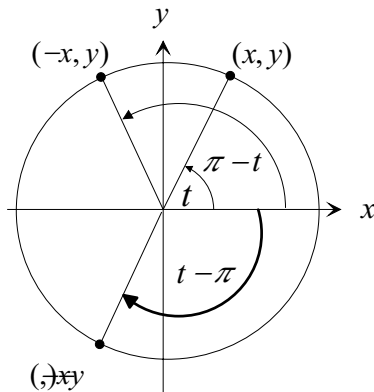
$$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}, \quad \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \tan \frac{3\pi}{4} = -1.$$

b) Note that $\frac{5\pi}{6} = \pi - \pi/6$ so the coordinates so the coordinates associated with this angle are $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Hence,

$$\sin \frac{5\pi}{6} = \frac{1}{2}, \quad \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \quad \text{and} \quad \tan \frac{5\pi}{6} = -\frac{1}{\sqrt{3}}.$$



The two symmetry properties discussed thus far are easily combined to obtain the coordinates of the point on the unit circle related to the angle $(t - \pi)$ rad. To see this introduce the notation $s = \pi - t$ and let (x, y) be the point on the unit circle determined by the angle t rad. Then $(-x, y)$ is associated with s rad. Using the symmetry property discussed at the beginning of this section, the point associated with $-s$ rad is obtained by simply negating the y -coordinate of that for s . (See [Figure 2.2](#).) This means that $(-x, -y)$ is determined by s . But $-s$ rad $= -(\pi - t)$ rad $= (t - \pi)$ rad. The figure below provides a complete visual account of this result.



Example 11 Using the symmetry of the circle and the information in [Table 2.2](#) determine the values of the sine, cosine, and tangent functions at a) $-\frac{2\pi}{3}$ rad and b) $-\frac{5\pi}{6}$ rad.

Solution:

a) Since $-\frac{2\pi}{3} = \pi/3 - \pi$ and $\pi/3$ is associated with $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, the coordinates of the point that determines the angle $-\frac{2\pi}{3}$ rad are $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Consequently,

$$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \quad \cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \text{and} \quad \tan\left(-\frac{2\pi}{3}\right) = \sqrt{3}.$$

b) Since the coordinates of the point on the unit circle determined by $\frac{\pi}{6}$ are $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $-\frac{5\pi}{6} = \frac{\pi}{6} - \pi$, we have using the point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

$$\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \quad \cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \tan\left(-\frac{5\pi}{6}\right) = \frac{1}{\sqrt{3}}.$$

As illustrated in [Figure 2.4](#) the coordinates of the points on the unit circle that determines the angles t rad and $(t + \pi)$ rad are symmetric with respect to the origin since they differ by an angle of π rad (180°). (Note the angle of π rad indicated in the figure by the bold curved arrow that separates the two angles t rad and $(t + \pi)$ rad.) Given that the coordinates on the unit circle associated with the angle t rad are (x, y) , those determined by $(t + \pi)$ rad are $(-x, -y)$. This suggests the relations

$$\begin{aligned}\sin(t + \pi) &= -y = -\sin t & \text{and} \\ \cos(t + \pi) &= -x = -\cos t.\end{aligned}$$

Also, we can calculate

$$\tan(t + \pi) = \frac{\sin(t + \pi)}{\cos(t + \pi)} = \frac{-\sin t}{-\cos t} = \tan t$$

for any angle t . Again, the remaining three trigonometric functions satisfy similar identities.

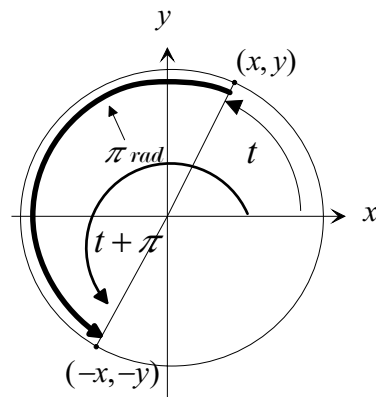


Figure 2.4: Symmetry properties of the angles t rad and $(\pi + t)$ rad.

Example 12 Determine the values of the sine, cosine, and tangent functions at $\frac{5\pi}{4}$ rad.

Solution: Consider *Figure 2.4* and recall that if the coordinates on the unit circle determined by the angle t rad are (x, y) , then the coordinates related to the angle $(t + \pi)$ rad are $(-x, -y)$. Since $\frac{5\pi}{4} = \frac{\pi}{4} + \pi$ and the point associated with $\frac{\pi}{4}$ rad is $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, the coordinates that determine the angle $\frac{5\pi}{4}$ rad are $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. The three values are

$$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \quad \cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \quad \text{and} \quad \tan \frac{5\pi}{4} = 1.$$

Example 13 Determine the value of the sine function at the angle at 240° .

Solution: Since $240^\circ = 60^\circ + 180^\circ$ and the point on the unit circle determined by the angle 60° is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ we see that $\sin 240^\circ = -\frac{\sqrt{3}}{2}$.

5. Periodic behavior of the trigonometric functions

The terminal sides of coterminal angles⁶ in standard position intersect the unit circle at a unique point P . Since the values of any trigonometric function depend only on the coordinates of P , it must be the same at coterminal angles. For example, since $\frac{\pi}{3} + 4\pi = \frac{13\pi}{3}$, the angles $\frac{\pi}{3}$ rad and $\frac{13\pi}{3}$ rad are coterminal so $\sin \frac{13\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. More generally, for any angle of measure t rad,

$$\sin(t + 2k\pi) = \sin t \quad (1)$$

where k can be any integer. The geometric symmetry of the unit circle suggests that the value 2π (or $k = 1$) is the smallest positive number for which Equation 1 holds. This property of the sine function is described by saying that it is **periodic** with **period** 2π . Similar statements hold for the cosine function so it is also periodic with period 2π . Furthermore, the secant and cosecant have period 2π since they are the reciprocals of the cosine and sine functions respectively⁷, The tangent and cotangent functions are also periodic but have period π as suggested by Exercise 5.

⁶Recall that coterminal angles have the same initial and terminal sides. Such angles must have radian measures that differ by an integer multiple of 2π .

⁷Recall that $\csc t = \frac{1}{y} = \frac{1}{\sin t}$, $\sec t = \frac{1}{x} = \frac{1}{\cos t}$, and $\tan t = \frac{y}{x} = \frac{1}{\cot t}$.

6. Exercises

EXERCISE 1. Determine the values of the trigonometric functions if they are defined at the angles: with radian measure

- (a) $-\pi$
- (b) $3\pi/2$
- (c) $-\pi/2$
- (d) $-\pi/4$
- (e) $2\pi/3$
- (f) $-5\pi/6$
- (g) $-\pi/3$
- (h) $3\pi/4$

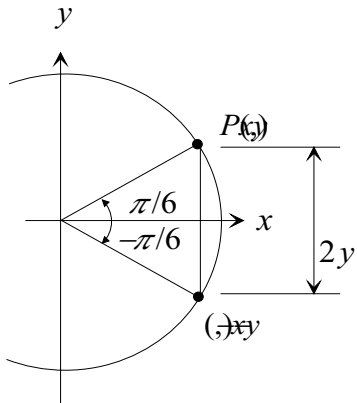
EXERCISE 2. Determine the values of the trigonometric functions at the angles with radian measure

- a) -4π b) $5\pi/2$ c) $-7\pi/2$ d) $-9\pi/4$ e) $5\pi/3$ f) $-11\pi/6$ g) $-4\pi/3$ h) $9\pi/4$.

EXERCISE 3. Determine the values of the trigonometric functions at the angles with radian measure

- a) 97π b) $15\pi/2$ c) $-31\pi/2$ d) $19\pi/4$ e) $9\pi/3$,f) $21\pi/6$ g) $14\pi/3$ h) $19\pi/4$.

EXERCISE 4. Find the coordinates of the point $P(x, y)$ on the unit circle determined by the angle $\pi/6$ rad. The figure below should provide some insight.



EXERCISE 5. Show that the period of the tangent and cotangent functions is π .

Solutions to Exercises

Exercise 1(a) Determine the values of the trigonometric functions at the angles: with radian measure $-\pi$.

Solution: Since $-\pi = 0 - \pi$ and the coordinates of the point on the unit circle determined by 0 rad are $(1, 0)$, the coordinates of the point P determined by $-\pi$ are $(-1, 0)$. Of course, this observation follows directly by simply using a drawing to locate the point on the unit circle determined by the angle $-\pi$ rad and then observing that the coordinates of that point P are $(-1, 0)$. The table below list the desired values or a dash indicates that the function is undefined at $-\pi$ rad.

t	P	sin	cos	tan	cot	sec	csc
$-\pi$	$(-1, 0)$	0	-1	0	$-$	-1	$-$



Exercise 1(b) Determine the values of the trigonometric functions at the angles: with radian measure $3\pi/2$.

Solution: Since $\frac{3\pi}{2} = \frac{\pi}{2} + \pi$, the coordinates of the point P determined by the angle $\frac{3\pi}{2}$ rad are $(0, -1)$. The table below list the desired values or a dash indicates that the function is undefined at $3\pi/2$ rad.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
$3\pi/2$	$(0, -1)$	-1	0	$-$	0	$-$	-1

□

Exercise 1(c) Determine the values of the trigonometric functions at the angles: with radian measure $-\pi/2$.

Solution: Since $-\frac{\pi}{2} = \frac{\pi}{2} - \pi$, the coordinates of the point P determined by the angle $-\frac{\pi}{2}$ rad are $(0, -1)$. The table below list the desired values or a dash indicates that the function is undefined at $3\pi/2$ rad. Note that this answer is exactly the same as that for the angle $\frac{3\pi}{2}$ rad given in the previous part of this exercise. This is because the two angles are coterminal since they differ by 2π .

t	P	sin	cos	tan	cot	sec	csc
$-\pi/2$	$(0, -1)$	-1	0	$-$	0	$-$	-1



Exercise 1(d) Determine the values of the trigonometric functions at the angles: with radian measure $-\pi/4$.

Solution: The coordinates of the point P determined by the angle $-\frac{\pi}{4}$ rad are $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

The table below list the desired values.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
$-\pi/4$	$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$

□

Exercise 1(e) Determine the values of the trigonometric functions at the angles: with radian measure $2\pi/3$.

Solution: Since $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$, the coordinates of the point P determined by the angle $\frac{2\pi}{3}$ rad are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The table below list the desired values.

t	P	sin	cos	tan	cot	sec	csc
$2\pi/3$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{-1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$

□

Exercise 1(f) Determine the values of the trigonometric functions at the angles: with radian measure $-5\pi/6$.

Solution: Since $-\frac{5\pi}{6} = \frac{\pi}{6} - \pi$, the coordinates of the point P determined by the angle $-\frac{5\pi}{6}$ rad are $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$. The table below list the desired values.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
$-\frac{5\pi}{6}$	$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{-2}{\sqrt{3}}$	-2

□

Exercise 1(g) Determine the values of the trigonometric functions at the angles: with radian measure $-\pi/3$.

Solution: The table below list the desired values.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
$-\pi/3$	$\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$\frac{-1}{\sqrt{3}}$	2	$\frac{-2}{\sqrt{3}}$

□

Exercise 1(h) Determine the values of the trigonometric functions at the angles: with radian measure $3\pi/4$.

Solution: The table below list the desired values.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
$3\pi/4$	$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$



Exercise 2. Determine the values of the trigonometric functions at the angles with radian measure

a) -4π b) $5\pi/2$ c) $-7\pi/2$ d) $-9\pi/4$ e) $5\pi/3$ f) $-11\pi/6$ g) $-4\pi/3$ h) $9\pi/4$.

Solution: The following table is derived using the values in [Table 2.2](#), the symmetry properties of the unit circle, and the periodic behavior of the trigonometric functions.

t	P	\sin	\cos	\tan	\cot	\sec	\csc
a) -4π	$(1, 0)$	0	1	0	—	1	—
b) $5\pi/2$	$(0, 1)$	1	0	—	0	—	1
c) $-7\pi/2$	$(0, 1)$	1	0	—	0	—	1
d) $-9\pi/4$	$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
e) $5\pi/3$	$\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$\frac{-1}{\sqrt{3}}$	2	$\frac{-2}{\sqrt{3}}$
f) $-11\pi/6$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
g) $-4\pi/3$	$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{-1}{2}$	$-\sqrt{3}$	$\frac{-1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$
h) $9\pi/4$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$

Exercise 2

Exercise 3. Determine the values of the trigonometric functions at the angles with radian measure

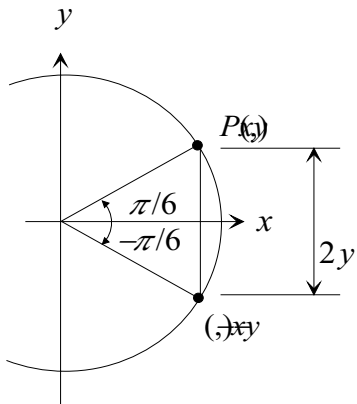
a) 97π b) $15\pi/2$ c) $-31\pi/2$ d) $19\pi/4$ e) $9\pi/3$,f) $21\pi/6$ g) $14\pi/3$ h) $19\pi/4$.

Solution: The following table is derived using the values in [Table 2.2](#), the symmetry properties of the unit circle, and the periodic behavior of the trigonometric functions.

t	P	\sin	\cos	\tan	\cot	\sec	\csc	Notes
a) 97π	$(-1, 0)$	0	-1	0	$-$	-1	$-$	$97\pi = 2(48)\pi + \pi$
b) $15\pi/2$	$(0, -1)$	-1	0	$-$	0	$-$	-1	$\frac{15\pi}{2} = 2(3)\pi + \frac{3\pi}{2}$
c) $-31\pi/2$	$(0, 1)$	1	0	$-$	0	$-$	1	$\frac{-31\pi}{2} = -14\pi + \frac{-3\pi}{2}$
d) $19\pi/4$	$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$	$\frac{19\pi}{4} = 2(2)\pi + \frac{3\pi}{4}$
e) $9\pi/3$	$(-1, 0)$	0	-1	0	$-$	-1	$-$	$\frac{9\pi}{3} = 2\pi + \pi$
f) $21\pi/6$	$(0, -1)$	-1	0	$-$	0	$-$	-1	$\frac{21\pi}{6} = 2\pi + \frac{3\pi}{2}$
g) $14\pi/3$	$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{-1}{2}$	$-\sqrt{3}$	$\frac{-1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$\frac{14\pi}{3} = 2(2)\pi + \frac{2\pi}{3}$
h) $-19\pi/4$	$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$	$\frac{-19\pi}{4} = -4\pi + \frac{-3\pi}{4}$

Exercise 4. Find the coordinates of the point $P(x, y)$ on the unit circle determined by the angle $\pi/6$ rad.

Solution: Consider the triangle in the figure below formed by the points $(\pm x, \pm y)$ and the origin. The two sides formed by radii of the unit circle ensure that this triangle is isosceles. Hence, this is an equilateral triangle since the angle at the origin has measure $\pi/3$ rad and the remaining two angles must be equal and, consequently, have radian measure $\pi/3$.



Since the triangle has one side of length one, all three sides must be one unit long.

Since the vertical side has length one we have

$$2y = 1 \quad \text{or} \quad y = \frac{1}{2}.$$

Further, since (x, y) in the figure is a point on the unit circle, we know that $x^2 + y^2 = 1$. Substituting $y = \frac{1}{2}$ into this equation gives

$$x^2 + \left(\frac{1}{2}\right)^2 = 1 \implies x^2 = \frac{3}{4}.$$

Since x must be positive it follows that

$$x = \frac{\sqrt{3}}{2}.$$

We have

$$P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Exercise 4

Exercise 5. Show that the period of the tangent and cotangent function is π .

Solution: We have already seen that $\tan(t + \pi) = \tan t$ for all numbers t in the domain of the tangent function. It remains to show that π is the smallest such number. Suppose there is a number c such that $\tan(t + c) = \tan t$ for all t in the domain of the tangent function. Then setting $t = 0$ yields

$$\tan 0 = \tan c$$

suggesting that c is an integer multiple of π . It follows that π is the smallest positive number for which $\tan(t + \pi) = \tan t$ for all numbers t in its domain. Since the $\cot t = \frac{1}{\tan t}$ it follows that the cotangent function also has period π . Exercise 5