LESSON 3:
RIGHT TRIANGLE TRIGONOMETRY

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1. Introduction

In Lesson 2 the six trigonometric functions were defined using angles determined by points on the unit circle. This is frequently referred to as the **circular definition** of the trigonometric functions. The next section (The triangular definition) of this lesson presents the right triangular definitions of the trigonometric functions. This raises a question about the consistency or agreement of the circular and right triangle definitions. For example, does the sine function produce the same value for a given angle regardless of the definition used. Predictably, the answer is affirmative as verified in section 3. The next two sections (section 4 and section 5) demonstrate how the trigonometric functions can be applied to arbitrary triangles. In Section 6 the circular definition of the trigonometric functions is extended to circles of arbitrary radii.

Remark 1 *Some examples and exercises in this lesson require the use of a calculator equipped with the **inverse trigonometric functions**. The inverse sine, cosine and tangent function keys are usually denoted on keypads by* $\sin^{-1}$, $\cos^{-1}$, *and* $\tan^{-1}$. *These functions are used to determine the measure of an angle* α *if* $\sin\alpha$ *(or* $\cos\alpha$ *or* $\tan\alpha$*) is known. For example, if* $\sin\alpha = .2$, *then* $\alpha = \sin^{-1}.2 = 11.537\,°$. *Before executing these functions the reader should ensure that the calculator is in the correct mode. Degree mode is used throughout this lesson. It should be noted that the general theory of inverse functions is rather sophisticated and presently lies beyond the scope of this tutorial.*

2. The triangular definition

Consider the right triangle in Figure 3.1 where α denotes one of the two non-right angles. The side of the triangle opposite the right angle is called the **hypotenuse**. The remaining two sides of the triangle can be uniquely identified by relating them to the angle α as follows. The **adjacent side** (or the **side adjacent** α) refers to the side that, along with the the hypotenuse, forms the angle α. The third side of the triangle is called the **opposite side** (or the **side opposite** α). The dependence of these labels on α is crucial since, for example, the side *opposite* α is *adjacent* to the other non-right angle of the triangle. The abbreviations **'hyp'** for the length of the hypotenuse, and **'opp'** and **'adj'** for the lengths of the opposite and adjacent sides respectively are used to define the values of the six trigonometric functions for the angle $\alpha \neq 90°$. These definitions are given in Table 3.1.

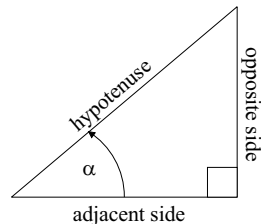Figure 3.1: Right triangle

$$\sin \alpha = \frac{\textbf{opp}}{\textbf{hyp}} \quad \cos \alpha = \frac{\textbf{adj}}{\textbf{hyp}} \quad \tan \alpha = \frac{\textbf{opp}}{\textbf{adj}}$$

$$\cot \alpha = \frac{\textbf{adj}}{\textbf{opp}} \quad \sec \alpha = \frac{\textbf{hyp}}{\textbf{adj}} \quad \csc \alpha = \frac{\textbf{hyp}}{\textbf{opp}}$$

Table 3.1: Definition of the trigonometric functions from a right triangle.

Example 2 *A right triangle with angle $\alpha \neq 90°$ has an adjacent side 4 units long and a hypotenuse 5 units long. Determine $\sin \alpha$ and $\tan \alpha$. Also, determine $\sin \beta$ and $\tan \beta$ where β denotes the second non-right angle of the triangle. Finally, use a calculator to determine α and β.*

Solution: *By the Pythagorean Theorem[1] $5^2 = 4^2 + (\boldsymbol{opp})^2$, so the length of the side opposite α is $3 = \sqrt{25 - 16}$ units. Consequently,*

$$\sin \alpha = \frac{\boldsymbol{opp}}{\boldsymbol{hyp}} = \frac{3}{5} \quad and \quad \tan \alpha = \frac{\boldsymbol{opp}}{\boldsymbol{adj}} = \frac{3}{4}.$$

Using a calculator (in degree mode) we find

$$\alpha = \sin^{-1} \frac{3}{5} \simeq 36.87°.$$

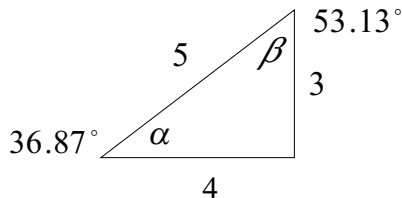*For the angle β, the roles of adjacent and opposite sides must be reversed. Hence,*

$$\sin \beta = \frac{4}{5} \quad and \quad \tan \beta = \frac{4}{3}.$$

Again, using a calculator we find

$$\beta = \sin^{-1} 4/5 \simeq 53.13°.$$

A sketch of the triangle is given below.

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[1] The lengths of the sides of a triangle satisfy $(\boldsymbol{adj})^2 + (\boldsymbol{opp})^2 = (\boldsymbol{hyp})^2$.

The reader should observe that

$$53.13\,^{\circ} + 36.87\,^{\circ} = 90.0\,^{\circ}$$

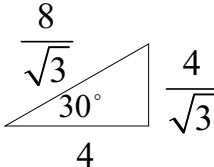*which serves as a partial confirmation,[2] but not a guarantee, of the correctness of the above calculations.*

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[2]The values of the three interior angles of a triangle sum to $180\,^{\circ}$. Hence, the two non-right angles of a right triangle must sum to $90\,^{\circ}$.

Example 3 *A right triangle with angle $\alpha = 30°$ has an adjacent side 4 units long. Determine the lengths of the hypotenuse and side opposite α.*

Solution: *The definition* $\cos \alpha = \dfrac{\boldsymbol{adj}}{\boldsymbol{hyp}}$ *suggests that* $\cos 30° = \dfrac{4}{\boldsymbol{hyp}}$. *So,*

$$\boldsymbol{hyp} = \frac{4}{\cos 30°} = \frac{4}{\sqrt{3}/2} = \frac{8}{\sqrt{3}}.$$

Similarly, $\sin 30° = \dfrac{\boldsymbol{opp}}{\boldsymbol{hyp}}$ *so the length of the side opposite α is*

$$\boldsymbol{opp} = \boldsymbol{hyp} \sin 30° = \left(\frac{8}{\sqrt{3}}\right)\left(\frac{1}{2}\right) = \frac{4}{\sqrt{3}}.$$

The Pythagorean Theorem provides a quick endorsement of the computed values. Specifically, note that

$$4^2 + \left(\frac{4}{\sqrt{3}}\right)^2 = 16 + \frac{16}{3} = 16\left(1 + \frac{1}{3}\right) = \frac{64}{3} = \left(\frac{8}{\sqrt{3}}\right)^2.$$

3. Consistency of the triangular definition

As illustrated in Figure 3.2, several right triangles may contain the same angle α. Triangles with the same angles but different side lengths are called **similar**. Similar triangles, then, have the same shape and differ only in size. This raises an immediate concern about using the definitions in Table 3.1. Specifically, do the values $\sin \alpha$, $\cos \alpha$, and the remaining trigonometric functions change with the size of the triangle? The answer is no as verified by the following argument. Since the two right triangles in Figure 3.2 are similar, geometric considerations ensure that the ratios of corresponding sides of the triangles satisfy

Figure 3.2: Similar right triangles.

$$\frac{O}{H} = \frac{o}{h}, \quad \frac{A}{H} = \frac{a}{h}, \quad \text{and} \quad \frac{o}{a} = \frac{O}{A}.$$

These equalities and the definitions in Table 3.1 suggest that the values of the trigonometric functions for the angle α are independent of the the triangle used to obtain them. For example

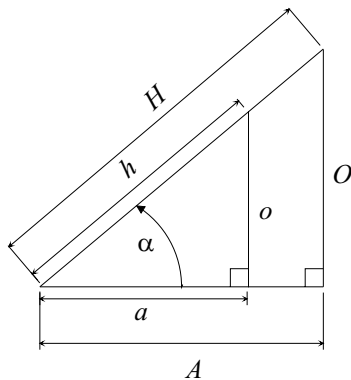$$\sin \alpha = \frac{O}{H} = \frac{o}{h}.$$

Also, using the larger triangle we have

$$\tan \alpha = \frac{O/H}{A/H} = \frac{O}{A},$$

while the smaller triangle suggests

$$\tan \alpha = \frac{o/h}{a/h} = \frac{o}{a}.$$

Since $\frac{o}{a} = \frac{O}{A}$, $\tan \alpha$ remains unchanged as the size, but not the shape, of the triangle fluctuates. Similar reasoning verifies the consistency of the triangle definitions of all the trigonometric functions.

Example 4 *Consider the right triangle in Figure 3.3. Find the length of the side adjacent to α if $\sin \alpha = 3/5$.*
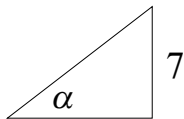

Figure 3.3: A right triangle.

Solution: *There is no way to compute the length of the adjacent side directly so we first compute the length of the hypotenuse. From Figure 3.3 and the triangular definition of the sine function we have*

$$\sin \alpha = \frac{\boldsymbol{opp}}{\boldsymbol{hyp}} = \frac{7}{\boldsymbol{hyp}}.$$

Since $\sin \alpha = \dfrac{3}{5}$ we have

$$\frac{7}{\boldsymbol{hyp}} = \frac{3}{5}$$
$$\implies \boldsymbol{hyp} = \frac{35}{3}.$$

The Pythagorean Theorem can now be applied to determine the adjacent side as follows

$$\boldsymbol{adj} = \sqrt{\left(\frac{35}{3}\right)^2 - (7)^2} = \sqrt{\frac{784}{9}} = \frac{28}{3}.$$

Of course the fundamental properties of similar triangles could have been used to solve this problem. Because $\sin \alpha = 3/5$, the given triangle in Figure 3.3 is similar to the triangle with angle α, a hypotenuse of length 5, and side opposite α of length 3. The similarity of these triangles is illustrated in the figure below. The Pythagorean Theorem indicates that the side opposite α in the smaller triangle has length $\sqrt{25 - 9} = 4$.
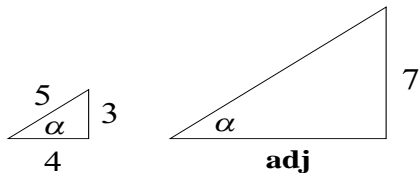
Because of this similarity we have the equality

$$\frac{\boldsymbol{adj}}{7} = \frac{4}{3}.$$

*Solving this equation gives*

$$\boldsymbol{adj} = \frac{28}{3}.$$

Example 5 *Determine the angles β and γ and the lengths of the sides a and c in the triangle Figure (a) below.*

Solution: *Construct the line segment h that is perpendicular to b connecting the angle β to the side b as indicated in Figure (b). Doing so forms two right triangles T_1 and T_2 with a common side h and base sides b_1 and b_2. Since $\sin 30° = h/4$ we have $h = 4\sin 30° = 2$. Also,*

$$\sin \gamma = \left(\frac{h}{2\sqrt{2}}\right) = \left(\frac{2}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}},$$

so $\gamma = 45°$. Since $\beta + 30° + 45° = 180°$, we have $\beta = 180° - 75° = 105°$. Finally,

$$b = b_1 + b_2 = 2\cot 30° + 2\cot 45° = 2(\sqrt{3} + 1) = 5.464\,1.$$
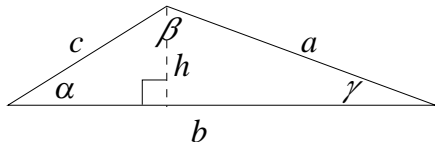

(a)   (b)
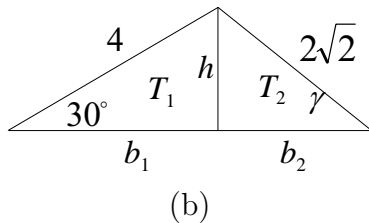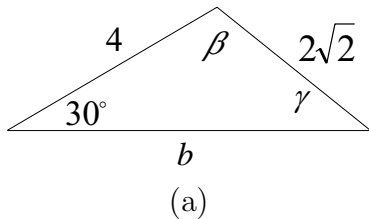
4. Law of Sines

Example 5 is suggestive of a general rule called the **Law of Sines**. Specifically, given the sample triangle in Figure 3.4 with sides a, b, and c opposite the angles α, β, and γ respectively, the Law of Sines states that

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}. \qquad (1)$$

Figure 3.4: Sample triangle.

To prove this consider the triangle in the figure below in which a line segment of length h is constructed perpendicular to side b connecting b to the angle β.

The two right triangles thus formed suggest that
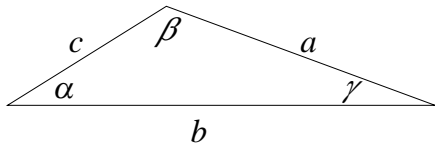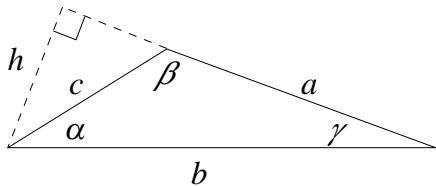
$$h = c \sin \alpha \ \text{ and } \ h = a \sin \gamma.$$

Hence,

$$c \sin \alpha = a \sin \gamma \implies \frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}. \tag{2}$$

A similar argument using the triangle below verifies that

$$\frac{\sin (180^\circ - \beta)}{b} = \frac{\sin \gamma}{c}.$$

Since $\sin \beta = \sin (180^\circ - \beta)$ we have

$$\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

Combining this equality with Equation 2 establishes Equation 1.

Example 6 *Reconsider Example 5. (The figure for that example is reproduced below.) The Law of Sines (Equation 1) facilitates the calculation of the remaining parts of the triangle since*

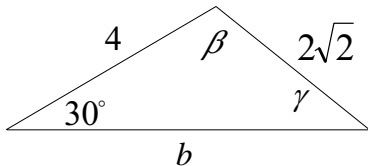$$\frac{\sin 30\,^\circ}{2\sqrt{2}} = \frac{\sin \gamma}{4}$$

so that

$$\sin \gamma = \frac{4(1/2)}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

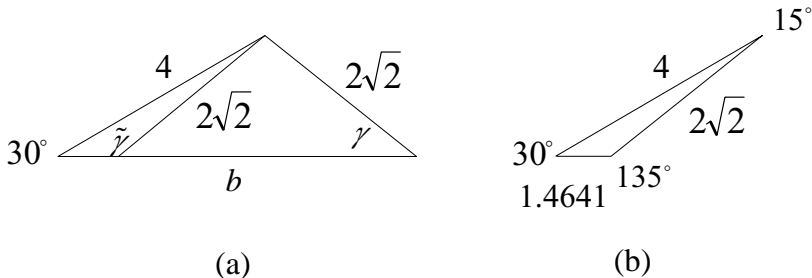*Hence, $\gamma = 45\,^\circ$. Also, $\beta = 180\,^\circ - 75\,^\circ = 105\,^\circ$ so that*

$$\frac{\sin 30\,^\circ}{2\sqrt{2}} = \frac{\sin 105\,^\circ}{b} \Longrightarrow b = 4\sqrt{2}\sin 105\,^\circ = 5.464\,1.$$

A calculator was used to compute $\sin 105\,^\circ$.

Example 7 *Actually, the conditions placed on the triangle in Example 6 permit two solutions when using the Law of Sines. Figure (a) below indicates that the angle $\gamma = 45\,^\circ$ in that example could be replaced with the angle*

$$\widetilde{\gamma} = (180\,^\circ - \gamma) = 180\,^\circ - 45\,^\circ = 135\,^\circ.$$

(a)   (b)

In this case

$$\widetilde{\beta} = 180\,^\circ - (135\,^\circ + 30\,^\circ) = 15\,^\circ.$$

Hence,

$$\widetilde{b} = 4\sqrt{2}\sin 15\,^\circ = 1.4641.$$

The second solution is depicted in Figure (b).

As the previous example illustrates, the Law of Sines does not always have a unique solution. Specifically, it is possible that two triangles possess a given angle and specified sides adjacent and opposite that angle. This is called the **ambiguous case** of the Law of Sines. There are rules for determining when the ambiguous case produces no solution, a unique solution, or two solutions. However, perhaps the best way of determining this is to simply solve the problem for as many solutions as possible. This strategy is illustrated in the exercises.

5. Law of Cosines

A second law that deserves attention is the **Law of Cosines** which is presented without justification. For a triangle with sides a, b, and c opposite the angles α, β, and γ respectively, the Law of Cosines states that

$$a^2 = b^2 + c^2 - 2bc \cos \alpha. \tag{3}$$

Observe that this rule reduces to the Pythagorean Theorem if α is a right angle since $\cos 90^\circ = 0$. This law is valid for any of the three angles of the triangle so it could have been stated as

$$b^2 = a^2 + c^2 - 2ac \cos \beta \ \text{ or } \ c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Example 8 *Suppose a triangle has adjacent sides of lengths* 2 *and* 3 *with an interior angle of measure* $\alpha = 70°$. *(See the figure below.) Then by the Law of Cosines the length of the side* a *opposite the angle* α *is given by*

$$a = \sqrt{2^2 + 3^2 - 2(2)(3)\cos 70°}$$
$$= \sqrt{2^2 + 3^2 - 4.104242}$$
$$= \sqrt{8.895758} = 2.982576.$$
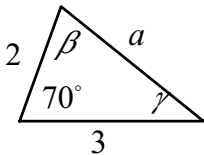
Observe that the Law of Cosines can be used to find β. *Indeed,*

$$\cos\beta = \frac{1}{2}\frac{-b^2 + a^2 + c^2}{ac}$$
$$= \frac{1}{2}\frac{-3^2 + 2.9826^2 + 2^2}{(2.982576)(2)}$$
$$= .326555$$

so that

$$\beta = \cos^{-1} .326555 = 70.9401\,^{\circ}.$$

Likewise, since $c^2 = a^2 + b^2 - 2ab \cos \gamma$, *we see that* $\cos(\gamma) = .776498$. *Hence,*

$$\gamma = \cos^{-1} .776498 = 39.0589\,^{\circ}.$$

As a check note that the sum of these three angles is $179.999\,^{\circ} \approx 180\,^{\circ}$. *Of course, the Law of Sines could also have been used to determine the measure of* β *and* γ. *For example, since*

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

we have

$$\frac{\sin 70\,^{\circ}}{2.982576} = \frac{\sin \beta}{3} \implies \sin \beta = \frac{3 \sin 70\,^{\circ}}{2.982576} = .9451822.$$

Hence,

$$\beta = \sin^{-1}(.9451822) = 70.9409\,^{\circ}.$$

Note that this last answer is not exactly the same as that obtained for β using the Law of Cosines. This demonstrates some of the difficulties with numerical calculations.

6. The circular definition revisited

The observation that the values of trigonometric functions are independent of the size of the right triangle suggests that the definition given for these functions on the unit circle can be modified to include circles of arbitrary radii. Examination of the figure below indicates that the values of the functions are those given in the table. The sides of the right triangle in the figure satisfy $adj = x$, $opp = y$, and $hyp = r$. Note that if the circle is the unit circle so that $r = 1$, then these values reduce to those given in Table 2.1 in Lesson 2.
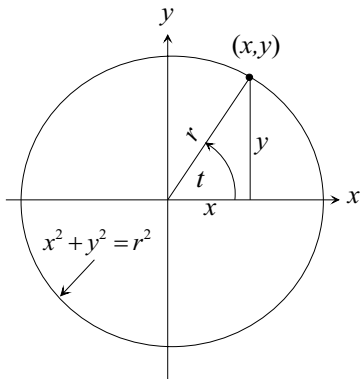
$$\sin t = \frac{y}{r} = \frac{\mathbf{opp}}{\mathbf{hyp}}, \quad \cos t = \frac{x}{r} = \frac{\mathbf{adj}}{\mathbf{hyp}},$$

$$\tan t = \frac{y}{x} = \frac{\mathbf{opp}}{\mathbf{adj}}, \quad \cot t = \frac{x}{y} = \frac{\mathbf{adj}}{\mathbf{opp}},$$

$$\sec t = \frac{r}{y} = \frac{\mathbf{hyp}}{\mathbf{adj}}, \quad \csc t = \frac{r}{x} = \frac{\mathbf{hyp}}{\mathbf{opp}}.$$
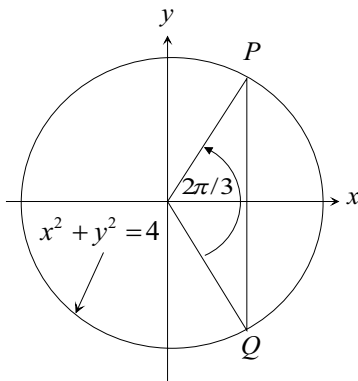
7. Exercises

EXERCISE 1. A right triangle contains a $35°$ angle that has an adjacent side of length 4.5 units. How long is the opposite side? How long is the hypotenuse? (You will need a calculator for this problem. Remember to set it to degree mode.)

EXERCISE 2. Suppose $\sin \alpha = 4/7$. Without using a calculator find $\cos \alpha$ and $\tan \alpha$.

EXERCISE 3. Let α denote a non-right angle of a right triangle. Prove that $\sin \alpha = \cos (90° - \alpha)$. Observe that a similar identity holds for the other five trigonometric functions.

EXERCISE 4. Determine the length of the chord $\overline{PQ}$ in the figure below.

EXERCISE 5. Let T be a triangle with sides a, b, and c opposite the angles α, β, and γ respectively as depicted in the figure below.

In each problem below determine the remaining parts of T if such a triangle exists. Remember that some conditions may permit two solutions. (See Example 7.)

(a) $a = 10$, $b = 7$, and $\alpha = 80\,^{\circ}$

(b) $a = 5$, $b = 7$, and $\alpha = 40\,^{\circ}$

(c) $a = 5$, $b = 7$, and $c = 10$

Solutions to Exercises

Exercise 1. A right triangle contains a $35\,^\circ$ angle that has an adjacent side of length 4.5 units. How long is the opposite side? How long is the hypotenuse? (You will need a calculator for this problem. Remember to set it to degree mode.)
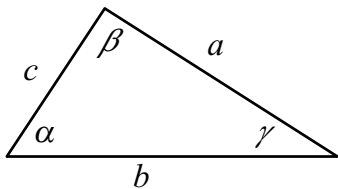
Solution: Let **opp** denote the length of the side opposite the $35\,^\circ$ angle. To find **opp**, use the formula $\tan \alpha = \frac{\textbf{opp}}{\textbf{adj}}$. (See Table 3.1.) Then

$$\tan 35\,^\circ = \frac{\textbf{opp}}{4.5}$$

so that

$$\textbf{opp} = 4.5 \tan 35\,^\circ \approx 3.150.$$

Let **hyp** denote the length of the hypotenuse. Then by the Pythagorean Theorem

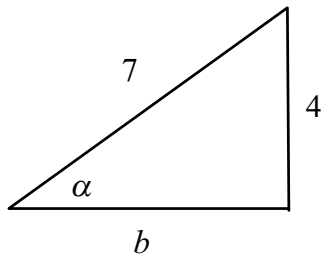$$\textbf{hyp} = \sqrt{3.150^2 + 4.5^2} = 5.49295.$$

As a simple check of these calculations we note that (in radian measure)

$$\sec 35\,^\circ = 1.22077 = \frac{5.49295}{4.5} = \frac{\textbf{adj}}{\textbf{hyp}}$$

Exercise 1

Exercise 2. Suppose $\sin \alpha = 4/7$. Without using a calculator find $\cos \alpha$ and $\tan \alpha$.
Solution: Since $\sin \alpha = 4/7$ we can construct the right triangle pictured below where the side opposite the angle α has length 4 and the hypotenuse has length 7.

The length b of the side adjacent α must satisfy

$$4^2 + b^2 = 7^2$$

from which it follows that $b = \sqrt{33}$. Appealing to Table 3.1 we see that
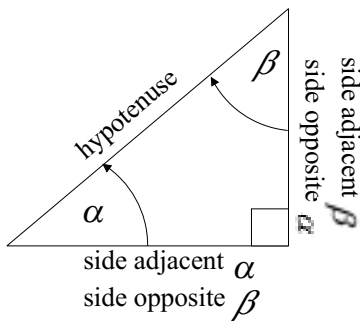
$$\cos \alpha = \frac{\sqrt{33}}{7} \quad \text{and} \quad \tan \alpha = \frac{4}{\sqrt{33}}.$$

<div align="right">Exercise 2</div>

Exercise 3. Prove that $\sin \alpha = \cos (90^\circ - \alpha)$
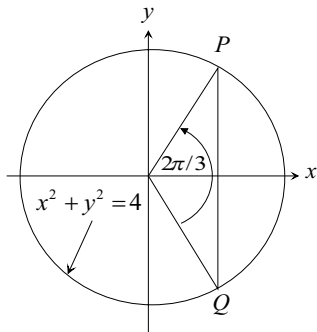
Solution: Consider the triangle below. Note that the side opposite α is adjacent to $\beta = (90^\circ - \alpha)$. Consequently,

$$\sin \alpha = \frac{\text{side opposite } \alpha}{\textbf{hyp}} = \frac{\text{side adjacent } (90^\circ - \alpha)}{\textbf{hyp}} = \cos(90^\circ - \alpha)$$

Exercise 3

Exercise 4. Determine the length of the chord $\overline{PQ}$ in the figure below.

Solution: The equation of the circle suggests that it has radius 2. Note the right triangle formed by the origin, the point P, and the x-axis. The angle of this triangle with vertex at the origin has radian measure $\pi/3$. Consequently,

$$\overline{PQ} = 2(\text{side opposite } \frac{\pi}{3}) = 2\left(2\sin\frac{\pi}{3}\right) = 2\sqrt{3}.$$

The Law of Sines could also be used to solve this problem. First, observe that the larger triangle formed by the origin and the points P and Q is isosceles so the

remaining angles have measure $\pi/6$. Then

$$\frac{\sin(2\pi/3)}{\overline{PQ}} = \frac{\sin(\pi/6)}{2} \implies \overline{PQ} = 4\sin(2\pi/3) = 2\sqrt{3}.$$

Finally, using the Law of Cosines we have

$$\overline{PQ} = \sqrt{8 - 8\cos(2\pi/3)} = \sqrt{8\left(1 + \frac{1}{2}\right)} = 2\sqrt{3}.$$

Exercise 4

Exercise 5(a) Let T be a triangle with sides $a = 10, b = 7$ and $\alpha = 80°$ where the angle α is opposite the side a. Determine the remaining parts of T if such a triangle exists.

Solution: By the Equation 1

$$\sin \beta = \frac{7 \sin 80°}{10} = .689365,$$

so

$$\beta = \sin^{-1} (.689365) = 43.579\,9°.$$

It follows that

$$\gamma = 180° - 80° - 43.579\,9° = 56.4201°.$$

Finally,

$$c = \frac{10 \sin 56.4201°}{\sin 80°} = 8.45967.$$

As a partial check of our calculations we examine how well our computed values satisfy

$$\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

(Why would we use these two ratios?) We have

$$\frac{\sin 43.5799\,^\circ}{7} = 9.84808 \times 10^{-2} \approx \frac{\sin 60.178\,^\circ}{8.8096}. \checkmark$$

A quick examination of the information given in this problem reveals that these conditions permit the ambiguous case of the Law of Sines. Evidently, then, there is a possibility of a second solution using

$$\widetilde{\beta} = 180\,^\circ - \beta = 180\,^\circ - 43.5799\,^\circ = 136.42\,^\circ.$$

However, this is impossible since we would then have

$$\alpha + \widetilde{\beta} > 180\,^\circ$$

which contradicts the fact the sum of all the angles of a triangle must be $180\,^\circ$. Hence, the given conditions provide the unique solution given above. $\square$

Exercise 5(b) Let T be a triangle with sides $a = 5, b = 7$ and $\alpha = 40\,^\circ$ where the angle α is opposite the side a. Determine the remaining parts of T if such a triangle exists.

Solution: Again, using Equation 1 we have

$$\sin \beta = \frac{7 \sin 40\,^\circ}{5} = .899903 \Longrightarrow \beta = \sin^{-1} .899903 = 64.145\,5\,^\circ$$

Hence,

$$\gamma = 180\,^\circ - 40\,^\circ - 64.145\,5\,^\circ = 75.8545\,^\circ.$$

Finally,

$$c = \frac{5 \sin 75.8545\,^\circ}{\sin 40\,^\circ} = 7.54276.$$

The relationships

$$\frac{\sin 64.145\,5\,^\circ}{7} = .128558 \approx \frac{\sin 75.8545\,^\circ}{7.54276} \checkmark$$

provides a reasonable check for the accuracy of these calculations.

The given conditions permit a second solution that arises from choosing

$$\widetilde{\beta} = 180\,^\circ - 64.145\,5\,^\circ = 115.855\,^\circ.$$

In this case

$$\widetilde{\gamma} = 180\,^{\circ} - 40\,^{\circ} - 115.855\,^{\circ} = 24.145\,^{\circ}$$

and

$$\widetilde{c} = \frac{5 \sin 24.145\,^{\circ}}{\sin 40\,^{\circ}} = 3.182.$$

As a quick check we examine

$$\frac{\sin 115.855\,^{\circ}}{7} = .1286 = \frac{\sin 24.145\,^{\circ}}{3.182}.\checkmark$$

□

Exercise 5(c) By the Equation 3

$$25 = 49 + 100 - 140 \cos \alpha$$

so

$$\alpha = \cos^{-1}\left(\frac{25 - 49 - 100}{-140}\right) = 27.66045\,^\circ.$$

Likewise,

$$49 = 25 + 100 - 100 \cos \beta \Longrightarrow \beta = 40.535802\,^\circ.$$

Finally, we have

$$100 = 25 + 49 - 70 \cos \gamma \Longrightarrow \gamma = 111.803759\,^\circ.$$

To check our computations we determine if the sum of the three computed angles is $180\,^\circ$:

$$27.66045\,^\circ + 40.535802\,^\circ + 111.803759\,^\circ \approx 180\,^\circ.\checkmark$$

□