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Department of Theoretical and Applied Mathematics

LESSON 7: TRIGONOMETRIC EQUATIONS

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Directory

Last Revision Date: August 16, 2001

Table of Contents

1. Introduction

A mathematical **equation** is an equality relationship involving mathematical quantities. For example, the expression

$$x^2 - x - 1 = 5 \tag{1}$$

is an equation. Equations may contain one or more variables. The above equation, for example, contains the variable x. Many scientific and technological endeavors require the **solution** of such equations. That is, it is necessary to determine the value(s) of a variable that make a given equation valid. This is called **solving** an equation. Equation 1 is easily solved using algebriac techniques as demonstrated in the remainder of this paragraph. First, consider

$$x^2 - x - 1 = 5 \Longrightarrow x^2 - x - 6 = 0 \Longrightarrow (x + 2)(x - 3) = 0.$$

Since the product of two numbers is zero only when one (or both) is, we see that $(x + 2)(x - 3) = 0$ only if $x + 2 = 0$ or $x - 3 = 0$. This means that Equation 1 has solution $x = -2, 3$. Sometimes a solution to an equation is written as a set. The **solution set** for Equation 1 is $\{-2, 3\}$.

A **trigonometric equation** is one that involves one or more trigonometric functions. For example,

$$\tan^2 t + 1 = \sec^2 t \tag{2}$$

is a trigonometric equation. The purpose of this lesson is to introduce the reader to the various techniques used to solve such equations. Equation 2 is solved in Example 2.

2. Solution of trigonometric equations

The basic strategy for solving a trigonometric equation is to use trigonometric identities and algebriac techniques to reduce the given equation to an equivalent but more manageable expression. For example, by first dividing by 2 and then using the trigonometric identity

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t,$$

the trigonometric equation

$$2\sin\left(\frac{\pi}{2} - t\right) = 1 \tag{3}$$

reduces to

$$\cos t = 1/2.$$

This last equation is more tractable than Equation 3. Indeed, since

$$\cos\left(\pm\frac{\pi}{3}\right) = \frac{1}{2}$$

its solution

$$t = \pm\frac{\pi}{3} + 2k\pi$$

where k is any integer. The addition of $2k\pi$ is necessary because of the periodic behavior of the cosine function. It follows that the solution set for Equation 3 is

$$A := \left\{t = \pm\frac{\pi}{3} + 2k\pi : k \in \mathbb{Z}\right\}$$

where $\mathbb{Z}$ denotes the set of integers.

The reader should note three things. First, the solution to Equation 3 required a knowledge of the cosine function at the angles $\pm\pi/3$. The values of the six trigonometric functions at such *special* angles was presented in section 4 of Lesson 2. A quick review of that material may be in order before continuing. Second, it is not uncommon for trigonometric equations to have infinitely many solutions. As with the above example, the periodic behavior of the trigonometric functions often ensures an infinite solution set. Finally, the choice of $\pm\frac{\pi}{3}$ as the fundamental solution to the given equation was a personal preference. The solution could also have been written as $t = \frac{\pi}{3} + 2k\pi$ or $\frac{5\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$, from which we get the set

$$B := \left\{ t = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \frac{5\pi}{3} + 2k\pi : k \in \mathbb{Z} \right\}.$$

Of course, the two sets A and B are equal. The reader should verify of this. In general, any two correct representations of the solution of a trigonometric equation must permit precisely the same values. Equivalently, in terms of sets, any two representations of the solution set for an equation must contain the same elements.

Sometimes the number of solutions is limited by the choice of the domain for the variable as demonstrated in the following example.

Example 1 *Find all solutions of the equation* $\tan t = \sin t$ *on the interval* $[0, 2\pi]$.

Solution: First, use the identity $\tan t = \dfrac{\sin t}{\cos t}$ *to arrive at the equivalent equation*

$$\frac{\sin t}{\cos t} - \sin t = 0.$$

Factoring out $\sin t$ *yields*

$$\sin t \left(\frac{1}{\cos t} - 1 \right) = 0.$$

Using the identity $\sec t = \dfrac{1}{\cos t}$ *suggests*

$$\sin t\,(\sec t - 1) = 0.$$

Since the product of two numbers is zero only when one of them is, either $\sin t = 0$ *or* $\sec t = 1$. *The sine function is zero on* $[0, 2\pi]$ *when* $t = 0, \pi$ *or* 2π. *Also,* $\sec t = 1$, *on* $[0, 2\pi]$ *when* $t = 0$, *a value for* t *that has already been realized. Therefore,* $\tan t = \sin t$ *on* $[0, 2\pi]$ *precisely when* $t = 0, \pi$ *or* 2π.

When solving equations involving two or more trigonometric functions it is often helpful to rewrite these in terms of one function.[1] This procedure is demonstrated in the following problem.

Example 2 *Find all solutions to* $\tan t + \sqrt{3} = \sec t$.

Solution: The identity $\tan^2 t + 1 = \sec^2 t$ *(See Table 6.3.) converts the secant function into an equivalent expression involving the tangent function. In an effort to exploit this identity square both sides*[2] *of the given equation to obtain*

$$\tan^2 t + 2\sqrt{3}\tan t + 3 = \sec^2 t.$$

Replacing $\sec^2 t$ *with* $1 + \tan^2 t$ *gives rise to*

$$\begin{aligned} \tan^2 t + 2\sqrt{3}\tan t + 3 &= 1 + \tan^2 t \\ \Longrightarrow 2\sqrt{3}\tan t + 3 &= 1 \\ \Longrightarrow \tan t &= -1/\sqrt{3}. \end{aligned}$$

A particular solution to this equation is $t = -\frac{\pi}{6}$*. Since the period of the tangent function is* π*, the original equation has solution* $t = -\frac{\pi}{6} + k\pi$ *for any integer* k.

[1] This is not always the case. For example, see Exercise 6.

[2] Squaring both sides of an equation may produce **extraneous** solutions. For example, $x = 2$ has the one solution 2. Squaring both sides of this equation yields $x^2 = 4$ which has the two solutions -2 and 2. (Also see Exercise 7.)

Example 3 *To find all solutions of*

$$\sin(2t) + \sin t = 0 \tag{4}$$

use the identity $\sin(2t) = 2\sin t\cos t$ *to obtain*

$$2\sin t\cos t + \sin t = 0.$$

Factoring out $\sin t$ *gives*

$$\sin t(2\cos t + 1) = 0.$$

Thus, $\sin t = 0$ *or* $\cos t = -\frac{1}{2}$*. Recall that* $\sin t = 0$ *when* $t = 0 + k\pi$ *and* $\cos t = -\frac{1}{2}$*, when* $t = \pm\frac{2\pi}{3} + 2k\pi$*. As usual,* k *denotes any integer. The solution to Equation 4 is, then,*

$$t = 0 + k\pi \quad \textit{or} \quad \pm\frac{2\pi}{3} + 2k\pi$$

where k *represents any integer. Consequently, the solution set for Equation 4 could be written as*

$$\{0 + k\pi : k \in \mathbb{Z}\} \cup \left\{\pm\frac{2\pi}{3} + 2k\pi : k \in \mathbb{Z}\right\}.$$

Example 4 *Find all solutions to*

$$4 \sin t \cos t = \sqrt{3} \tag{5}$$

on the interval $[0, \pi)$*.*
Solution: Rewrite Equation 5 as

$$2(2 \sin t \cos t) = \sqrt{3}.$$

Dividing this last equation through by 2 *and using the double angle formula*

$$\sin(2t) = 2 \sin t \cos t$$

yields

$$\sin(2t) = \frac{\sqrt{3}}{2}.$$

This means $2t = \frac{\pi}{3} + 2k\pi$ *or* $2t = \frac{2\pi}{3} + 2k\pi$ *where* k *can be any integer. Consequently,*

$$t = \frac{\pi}{6} + k\pi \quad \textit{or} \quad t = \frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z},$$

describes all solutions to Equation 5. An appropriate representation for the solution set to this equation is

$$\left\{ \frac{m\pi}{6} + k\pi : m = 1, 2 \ \textit{and} \ k \in \mathbb{Z} \right\}.$$

Example 5 *Determine the values of* $\alpha \in [0^\circ, 360^\circ)$ *for which*

$$\cos\alpha = \sin\frac{\alpha}{2}.$$

Solution: By the half angle formula for the sine function the given equation reduces to

$$\cos\alpha = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

Squaring both sides of this equation and then multiplying by 2 suggests that

$$2\cos^2\alpha = 1 - \cos\alpha,$$

or

$$2\cos^2\alpha + \cos\alpha - 1 = 0.$$

Factoring produces

$$(2\cos\alpha - 1)(\cos\alpha + 1) = 0$$

so $\cos\alpha = 1/2$ *or* $\cos\alpha = -1$. *The values of* $\alpha \in [0^\circ, 360^\circ)$ *that satisfy one or the other of these last two relations are* $60^\circ, 300^\circ,$ *or* 180°.

3. Exercises

EXERCISE 1. Find all values of t for which $\sin^2 t - 1 = 0$.

EXERCISE 2. Find the solution set for the equation $\cot t = \cos t$ on the interval $[0, 2\pi]$.

EXERCISE 3. Find all values of t for which $\tan^2 t - 1 = 0$.

EXERCISE 4. Find all values of t for which $\sin^2 t = 2\sin t - 1$.

EXERCISE 5. Find all values of t for which $\dfrac{(1-\sqrt{3})\tan t}{\sqrt{3}-\tan^2 t} = 1$.

EXERCISE 6. Determine the values of $\alpha \in [0\,^\circ, 360\,^\circ)$ for which $\cos(2\alpha) = 2\sin\alpha\cos\alpha$.

EXERCISE 7. Determine the values of $\alpha \in [0\,^\circ, 360\,^\circ)$ for which $\sin\alpha = \cos\frac{\alpha}{2}$.

EXERCISE 8. Find the solution set for the equation $\cos(\pi + t) = \sin(\pi - t)$.

Solutions to Exercises

Exercise 1. Find all values of t for which $\sin^2 t - 1 = 0$.
Solution: Rewrite the given equation as

$$\sin^2 t = 1$$

and take the square root of both sides to obtain

$$\sin t = \pm 1.$$

This equation is satisfied at odd multiples of $\frac{\pi}{2}$ resulting in the solution

$$t = \frac{(2k-1)\pi}{2}$$

where k is any integer.

An alternate but similar solution can be obtained by factoring the expression on the left of the given equation to obtain $\sin^2 t - 1 = (\sin t + 1)(\sin t - 1)$. Thus, the problem reduces to solving $(\sin t + 1)(\sin t - 1) = 0$. This means that either $(\sin t + 1) = 0$ or $(\sin t - 1) = 0$. Once again $\sin t = \pm 1$.

Exercise 1

Exercise 2. Find the solution set for the equation $\cot t = \cos t$ on the interval $[0, 2\pi]$.

Solution: First, use the identity $\cot t = \dfrac{\cos t}{\sin t}$ to arrive at the equation

$$\frac{\cos t}{\sin t} - \cos t = 0.$$

Factoring out $\cos t$ yields

$$\cos t \left(\frac{1}{\sin t} - 1 \right) = 0.$$

Using the identity $\csc t = \dfrac{1}{\sin t}$ suggests

$$\cos t \, (\csc t - 1) = 0.$$

Since the product of two numbers is zero only when one of them is, either $\cos t = 0$ or $\csc t = 1$. The cosine function is zero on $[0, 2\pi]$ when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, $\csc t = 1$, on $[0, 2\pi]$ when $t = \frac{\pi}{2}$, a value for t that has already been realized. Therefore, the solution set for the given equation is

$$\left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

Exercise 2

Exercise 3. Find all values of t for which $\tan^2 t - 1 = 0$.
Solution: Write $\tan^2 t - 1 = 0$ as $\tan^2 t = 1$ and take the square root of both sides to obtain $\tan t = \pm 1$. This last equation is satisfied at odd multiples of $\frac{\pi}{4}$ resulting in the solution

$$t = \frac{(2k-1)\pi}{4}$$

where k is any integer. Exercise 3

Exercise 4. Find all values of t for which $\sin^2 t = 2\sin t - 1$.
Solution: Collecting all terms on one side of the given equation reveals that it is quadratic in $\sin t$. Specifically,

$$\sin^2 t - 2\sin t + 1 = 0.$$

Factoring this last equation results in $(\sin t - 1)^2 = 0$ or $\sin t = 1$. The solution is

$$t = \frac{\pi}{2} + 2k\pi$$

for any integer k. Exercise 4

Exercise 5. Find all values of t for which $\dfrac{(1-\sqrt{3})\tan t}{\sqrt{3}-\tan^2 t} = 1.$

Solution: At first glance the equation looks rather foreboding, but multiplying both sides by $\sqrt{3}-\tan^2 t$ and collecting terms onto one side rids the equation of its imposing denominator. Doing so we obtain

$$(1-\sqrt{3})\tan t = \sqrt{3}-\tan^2 t,$$

which can be rewritten as the quadratic type equation

$$\tan^2 t + (1-\sqrt{3})\tan t - \sqrt{3} = 0.$$

This last expression can be factored to obtain

$$(\tan t + 1)\left(\tan t - \sqrt{3}\right) = 0.$$

Evidently then, either $\tan t = \sqrt{3}$ or $\tan t = -1$. Particular solutions to these two equations are $t = -\frac{\pi}{4}$ and $t = \frac{\pi}{3}$. Since the period of the tangent function is π, it follows that the general solution to the given problem is

$$t = \frac{\pi}{3} + k\pi, -\frac{\pi}{4} + k\pi$$

for any integer k.

Exercise 5

Exercise 6. Determine the values of $\alpha \in [0\,^\circ, 360\,^\circ)$ for which $\cos(2\alpha) = 2\sin\alpha\cos\alpha$.
Solution: Use the identity

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

to rewrite the given equation as

$$\cos(2\alpha) = \sin(2\alpha). \tag{6}$$

This means that $2\alpha = 45\,^\circ$ or $225\,^\circ$ so $\alpha = 22.5\,^\circ$ or $112.5\,^\circ$.

Observe that in this case it was not necessary to rewrite Equation 6 in terms of one trigonometric function to solve the equation. However, this can be done in a way some users may prefer. To see this, divide Equation 6 through by $\cos{(2\alpha)}$ to obtain

$$\tan{(2\alpha)} = 1.$$

This last expression has the same solution as that given above for Equation 6.

Exercise 6

Exercise 7. Determine the values of $\alpha \in [0\,°, 360\,°)$ for which $\sin\alpha = \cos\frac{\alpha}{2}$.
Solution: Square both sides of the given equation and then use the identity

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

to obtain

$$\sin^2\alpha = \frac{1+\cos\alpha}{2}.$$

Next, applying the identity $\sin^2\alpha = 1 - \cos^2\alpha$ and multiplying through by 2 yields

$$2 - 2\cos^2\alpha = 1 + \cos\alpha,$$

or

$$2\cos^2\alpha + \cos\alpha - 1 = 0.$$

This last equation can be factored to obtain $(\cos\alpha + 1)(2\cos\alpha - 1) = 0$ so that $\cos\alpha = -1, \frac{1}{2}$. Consequently, $\alpha = 180\,°$ or $\alpha = 60\,°, 300\,°$. Since we squared both sides of the given equation some of these solutions may be extraneous. Checking each value in the original equation indicates that all three are valid. Exercise 7

Exercise 8. Find the solution set for the equation $\cos(\pi + t) = \sin(\pi - t)$.
Solution: Using the formulas for the sine and cosine of the sum of two angles the given equation reduces to

$$-\cos t = \sin t \tag{7}$$

Note that if $\cos t = 0$ then $\sin t \neq 0$ so we may assume that $\cos t \neq 0$. Dividing Equation 7 by $\cos t$ gives

$$\tan t = -1.$$

Hence, the solution set to $\cos(\pi + t) = \sin(\pi - t)$ can be represented by

$$\left\{-\frac{\pi}{4} + k\pi : k \in \mathbb{Z}\right\}.$$

Exercise 8