

An analysis of simulators and tracking systems for laser Doppler velocimeters

T. S. DURRANI, T. H. WILMSHURST

Department of Electronics, University of Southampton

C. GREATER

Fluid Dynamics Unit, Physics Department, University of Edinburgh

Received 31 July 1972

Analogue and digital techniques are proposed for the simulation of signals arising in laser Doppler velocimeters. Their merits are evaluated.

Of special interest is the estimation of the instantaneous frequency of the Doppler signal from the zero crossing rate. An analysis is presented whereby the biases introduced in this estimation method are derived for two optical configurations of the velocimeter. This analysis is extended to the case of additive Gaussian noise on the signal, and the cumulative effect of additive noise on the zero crossing rate is determined.

The so called "drop-out problem" which pervades the frequency estimation of the instantaneous Doppler signal is next considered, and expressions are derived for mean drop-out rate and the mean interval between drop-outs in terms of an arbitrary signal threshold. These should facilitate choice of threshold detection for given rms Doppler signal levels, and in achieving relative compromises in terms of signal bandwidth. Finally a novel form of frequency tracking system is discussed and its performance critically assessed.

List of symbols

$A_{1,2}$	Weighting constant for Doppler signal arising from different velocimeter configurations
E	Expectation operator
$\{H(t)\}$	Envelope of Doppler signal
H_0	Doppler signal threshold level
$K_{1,2}$	Bandwidth ratio
N_i	Total input additive noise power
$R(\tau)$	Autocorrelation function for lag τ
$W(t)$	Weighting function associated with velocimeter set-up
$p(.)$	Probability density function of random variable $(.)$
$x(t), y(t)$	Bandpass filtered Doppler signal
$\Phi(f)$	Power spectral density function at frequency f
ψ_i	Total input signal power
η_i	Input signal-to-noise ratio

λ	Zero crossing rate
$\rho(\tau)$	Normalized autocorrelation function for lag (τ)
μ	Normalized threshold level
$(\omega_0)f_0$	(Angular) central frequency of Doppler signal
$(\omega_1)f_1$	(Angular) central frequency of bandpass filter.

1. Introduction

One of the biggest problems in setting up a laser Doppler system for the measurement of fluid flow is in the signal analysis. Firstly there are the practical problems of constructing a system to track high frequencies where the signal has wide variations in amplitude and may have a large noise component, and secondly there are a number of fundamental properties of the Doppler signal which place limitations on the accuracy of measurement. These arise out of the fact that the spectrum associated with the passage of a single scattering centre across the measuring volume has a finite width, and have already been the subject of some discussion in the literature [1, 2]. Two cases arise, (a) where the density is high and it can be assumed that the mean distance between scatterers is negligibly small as compared to the measuring volume and (b) where the density of scattering centres is very small, the time interval between arrivals being of the same order as the transit time, so that there is seldom more than one scatterer in the measuring volume. Case (a) gives a signal which can be considered as a high density shot noise process and is the one which we will consider here. For case (b), special problems arise as the signal comes in bursts and there is a high probability of losing the signal altogether.

In this paper we will make some comparisons between different types of time domain analysing systems in the light of the experience gained by a number of people working on laser anemometry at Southampton University. We will also describe how various methods have been used for the simulation of Doppler signals. This is of particular practical importance as it is necessary to have some means of checking the performance of an analysing system in various situations and there seems no easy means of setting up a flow having known statistical fluctuations where the various parameters can be varied. It is of course possible to make comparisons with other measuring instruments such as hot wire anemometers and to compare results with the well developed theories of homogeneous turbulence, for say, flow behind a grid, and comparisons of this type have in fact already been made by the authors. In these cases, however, it is not clear at which stage errors are being introduced, e.g. whether they are being introduced by the optical or the signal analysing system.

Before proceeding with the discussion on trackers and simulators an outline of the statistics of laser Doppler signals is required.

2. Statistics of laser Doppler signals

For the case of high density shot noise there are two ways of writing the output $x(t)$ from the photodetector, both of which are equivalent to passing white noise through a narrow band filter of bandwidth $\Delta\omega$ and centre frequency ω_0 , the Doppler frequency. They are

$$(a) \quad x(t) = \int_{\tau=-\infty}^{\tau=\infty} W(t - \tau) \cos \omega_0 (t - \tau) dN(\tau) \quad (1)$$

where

$$E[dN(\tau)] = 0 \text{ for all}$$

and

$$E[dN(\tau)dN(\mu)] = d\tau \quad \tau = \mu \\ = 0 \quad \text{otherwise}$$

and

$$(b) \quad x(t) = \sum_{i=-\infty}^{\infty} W(t - T_i) \cos \omega_0 (t - T_i),$$

where in both cases $W(t)$ is the weighting function for the particular optical system [3].

The second of these formulations is the more significant physically especially for the case of low density shot noise where the interval between arrival times T_i is great, but from the point of view of use as a model for a simulator for the high density case, the first formulation is more useful.

Alternatively Equation 1 may be written as orthogonal components:

$$x(t) = a(t) \cos \omega_0 t + b(t) \sin \omega_0 t \\ = H(t) \cos (\omega_0 t - \phi(t)),$$

where

$$a(t) = \int_{-\infty}^{\infty} W(t - \tau) \cos \omega_0 \tau dN(\tau) \\ b(t) = \int_{-\infty}^{\infty} W(t - \tau) \sin \omega_0 \tau dN(\tau). \quad (2)$$

$$H^2(t) = a^2(t) + b^2(t)$$

is the squared amplitude, and

$$\phi(t) = \tan^{-1} b(t)/a(t)$$

is the instantaneous phase.

This process can most easily be visualized by reference to a phasor diagram (Fig. 1) where the observer is moving round at the Doppler frequency ω_0 . The phasor is then of

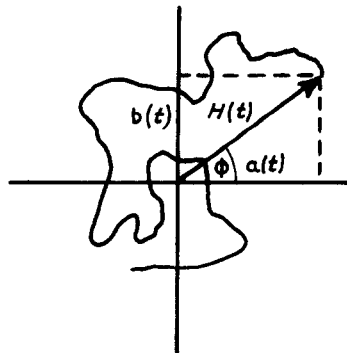


Figure 1 Phasor diagram with frame of reference rotating at constant angular velocity.

length H and moves in a random walk over the complex plane where the correlation time associated with the instantaneous frequency $(d\phi)/dt$ is of order $1/\Delta\omega$.

Consideration of this diagram brings out a number of points immediately. First of all although the rate of wandering is slower than the angular speed of observation, the mean rate of zero crossings will be greater than π/ω and so the counting of zero crossing rate gives a velocity measurement which is higher by a factor which increases as the ratio of bandwidth to centre frequency $\Delta\omega/\omega_0$. This fact was pointed out by Adrian [4]. It is also clear that if noise is present in the system the zero crossing rate is also artificially increased. Another factor which is of interest is the rate at which the envelope $H(t)$ fluctuates, as this determines what is known as the drop-out rate, i.e. the rate at which the signal drops below some preset threshold level.

These factors are of great importance as they affect both the design of signal analysing systems and the interpretation of results. In Section 4 some new theoretical results will be given for the statistics of zero crossings and for the statistics of the envelope fluctuations. It is difficult to verify these experimentally as variation of the various parameters independently is not possible in an experimental system, so we also describe the methods which have been used to simulate a laser Doppler system. Simulators which operate at the real time frequencies involved are also important, in that they can be used for testing the response of analysing systems to changes in modulation frequency, ambiguity bandwidth, etc.

3. Doppler signal simulation

Three distinct types of simulation, all based on the theoretical model described in Section 2, have been used. The first two of these involve analogue and digital computer techniques and were primarily used to study the form of the signal and to test the analytical results developed. The hybrid system can operate at the actual frequencies encountered in an experimental situation, i.e. a few MHz.

3.1. Analogue computer simulation

The problem here is to construct a narrow band filter whose centre frequency is voltage tuned and can be modulated with a given input signal corresponding to a velocity record. Passing white noise through this filter then gives the required Doppler signal. It is possible to use an analogue computer for this purpose but there are a number of quite serious problems. The first, which there seems no way of overcoming, is that as the filter is constructed from integrator units, its centre frequency cannot be varied at a rate greater than the bandwidth $\Delta\omega$, which is not a physical limitation in the real flow situation. Secondly, construction of filters whose frequency response corresponds closely to the Doppler spectrum, i.e. a Gaussian curve or a triangle, requires a very large number of components and there is normally a tendency for instability.

It was found in fact that a second order filter gave a signal possessing essentially all the Doppler characteristics and the patch layout for this is given in Fig. 2. The two integrators $A1$, $A2$ and the inverter $A3$ from the main filter loop, and the potentiometer $P3$ and multiplier M , control the bandwidth and centre frequency respectively. The potentiometer $P1$ and $P2$ are used for manual centre frequency adjustment. Fig. 3 shows a plot

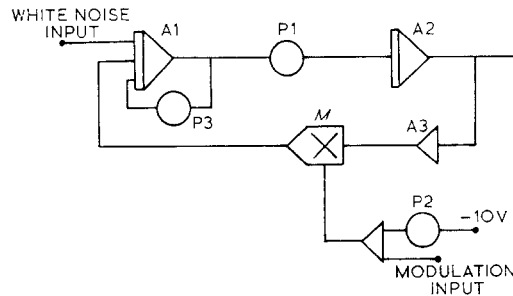


Figure 2 Patch layout for analogue simulator. *A1* and *A2* are integrators, *A3* is an inverter and *P1*, *P2* and *P3* are potentiometers. *M* is a multiplier for modulation of the centre frequency. Bandwidth adjustment is by *P3* and preset centre frequency adjustment by *P1* and *P2*.

obtained by passing the output from the simulator through a wave analyser and can be seen to be characteristic of the Doppler spectrum.

The analogue computer simulation method was found to be only partly satisfactory in that on the one hand the frequencies could not be made high enough to make it useful for testing data analysing systems, whilst on the other hand the characteristics of the Doppler spectrum could not be fitted sufficiently closely to make it a good means of testing the analytical theories. For this reason these two aspects were treated separately by hybrid and digital techniques respectively.

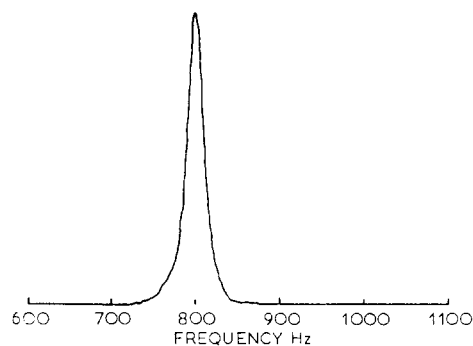


Figure 3 Frequency spectrum of output from simulator obtained by passing the output through a wave analyser.

3.2. Digital computer simulation

In essence the digital method again entails filtering of white noise, but now using digital filtering techniques. The great advantage here is that the filter characteristics can easily be matched exactly, i.e. apart from round off and discretization errors. The filtering is performed by convolution with the impulse response function for the filter and thus for a sequence of random numbers N_k , normally distributed with zero mean, the required output sequence X_k is

$$X_k = N_k * W_k \cos \omega_0 k.$$

*denotes discrete convolution. W_k is the discrete equivalent of the weighting function $W(t)$ in Equation 1. An alternative method of performing the simulation which is not so convenient to apply digitally but which forms the basis of the hybrid simulator, is to apply Equation 2, noting that $a(t)$ and $b(t)$ are in fact orthogonal components of the envelope and can themselves be generated by low pass filtering white noise where the bandwidth is $\Delta\omega/2$. The block diagram for the simulator is then as shown in Fig. 4. The point still remains that to use this as a practical system tester one would like to be able to modulate the centre frequency ω_0 , keeping the ratio $\Delta\omega/\omega_0$ constant. It was found that a rather neat way of doing this was to use a digital pseudo-random number generator. The particular one used [5] gives an output resembling low pass filtered white noise, the bandwidth being proportional to the clock input to the generator, which in this case is $\cos \omega_0 t$, the rms amplitude of the noise being constant. These are in fact exactly the requirements we need. Thus the input signal representing the velocity record controls the V.C.O. and hence sets both ω_0 and $\Delta\omega$.

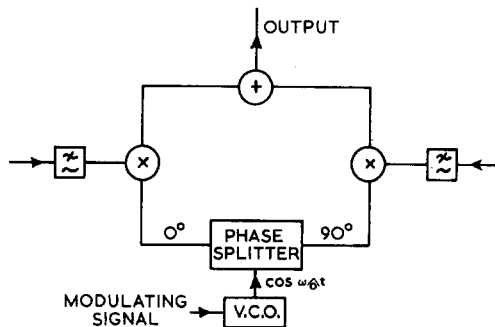


Figure 4 Hybrid simulator.

For the testing of data analysing systems this simulator has proved of great value. For testing analytical theories the ripple introduced by generating the filtered white noise from a discrete sequence of digits was found to be too great although in principle this difficulty can easily be overcome by increasing the number of registers in the digital section.

4. Zero crossing analysis

Intrinsically, signals arising in LDV systems carry Doppler information about the instantaneous velocity of the medium under observation. Processing of these signals is performed by some form of frequency detection technique. Most commonly, zero crossings of the Doppler signal are determined and the value of the instantaneous frequency derived by a frequency to voltage conversion.

Even under steady flow conditions, estimation of the mean Doppler frequency by zero crossing techniques leads to a bias [4] due to the finite width of the Doppler signal spectrum (obtained after bandpass filtering of the photodetector signal). Signals arising from two optical configurations of the velocimeter are considered here, and bias on the mean frequency – determined from the zero crossing rate – is calculated. The analysis is further extended to the case of Doppler signal immersed in broad band noise, and the mean zero crossing rate is evaluated.

I. Mask system [3]

For this case the power spectrum of the Doppler signal (after bandpass filtering) is given by

$$\Phi_1(f) = \psi_i \frac{3}{2A_1} \left[1 - \left| \frac{f - f_0}{A_1} \right| \right]^2 \quad f_0 - A_1 \leq f \leq f_0 + A_1$$

$$= 0 \quad \text{elsewhere} \quad (3)$$

where ψ_i = Signal power; f_0 = Centre frequency of the spectrum corresponding to a steady flow velocity, and A_1 = Weighting constant associated with the optical geometry [3]. This is related to the weighting function $\{W(t)\}$ of Equations 1 and 2.

The associated autocorrelation function of the Doppler signal is given by

$$R_1(\tau) = \psi_i \frac{6}{2\pi A_1 \tau} \left(1 - \frac{\sin 2\pi A_1 \tau}{2\pi A_1 \tau} \right) \cos 2\pi f_0 \tau. \quad (4)$$

II. Gaussian beam system [2-4]

Here the power spectrum of the Doppler signal is given by

$$\Phi_2(f) = \psi_i 4\sqrt{\pi\sigma^2} \exp \{ - (2\pi\sigma)^2 (f - f_0)^2 \} \quad 0 \leq f \leq \infty \quad (5)$$

with the associated correlation function

$$R_2(\tau) = \psi_i \exp \left(- \frac{\tau^2}{4\sigma^2} \right) \cos 2\pi f_0 \tau, \quad (6)$$

where

$$A_2 = \frac{1}{\sqrt{2\pi\sigma^2}} = \text{weighting constant},$$

and σ = parameter dependent on the optical geometry of the system [2, 3].

On the basis of the well-known analysis of Rice [6, 7] for narrow-band stationary, normally distributed processes, such as those implemented by Equations 3 or 5, the mean rate of zero crossings is given by

$$\lambda = \frac{1}{\pi} \left[- \frac{R''(0)}{R(0)} \right]^{\frac{1}{2}} \quad (7a)$$

$$= \frac{1}{\pi} \left[\frac{\int_0^{\infty} f^2 \Phi(f) df}{\int_0^{\infty} \Phi(f) df} \right]^{\frac{1}{2}}, \quad (7b)$$

$$\text{where } R''(\tau) = \frac{d^2 R(\tau)}{d\tau^2}.$$

I. For the Mask System, from Equations 3 or 4, using Equation 7

$$\lambda_1 = 2 \sqrt{f_0^2 + \frac{A_1^2}{10}}. \quad (8a)$$

II. For the Gaussian beam system, from Equations 5 or 6, and 7

$$\lambda_2 = 2 \sqrt{f_0^2 + \frac{A_2^2}{4\pi}}. \quad (8b)$$

Thus the respective biases are given by

$$b_1 = \lambda_1 - 2f_0 \simeq \frac{A_1^2}{10f_0}, \quad (9a)$$

$$b_2 = \lambda_2 - 2f_0 \simeq \frac{A_2^2}{4\pi f_0}. \quad (9b)$$

As explained in Section 2, the biases are proportional to the ratio of rms bandwidth to the centre frequency (f_0) of the Doppler signal. Thus signals with identical rms bandwidths arising from the two optical configurations would lead to exactly equal bias in the zero crossing rate, at the same centre frequency.

4.1. Effect of additive noise [8]

Analysis of the previous section was concerned with the idealized case of detection of the Doppler signal with no corrupting noise. However, in practice, noise is essentially introduced by three classes of noise sources: (a) optical, (b) fluid dynamic, (c) electronic system noise sources. Since the instantaneous Doppler signal reflects the true medium velocity, it is independent, and in general, uncorrelated with the noise introduced in the transmission and detection of the scattered radiation.

It is well established that the additive noise is broad band as compared to the Doppler signal bandwidth, and is, not unpredictably, white and Gaussian. Since photodetection of the scattered light is always followed by bandpass filtering, the overall signal at the output of the filter stage may be expressed as

$$y(t) = x(t) + n(t) \quad (10)$$

where $\{x(t)\}$ is the Doppler signal of Equations 1 or 2, and $\{n(t)\}$ represents the additive

noise component, and its frequency characteristics represent the bandpass filter response, as noise input to this filter is broad-band white noise.

Two types of symmetric bandpass filters are considered (symmetric about a central frequency f_1 say), and the consecutive effect of additive noise on the zero crossing rate of the detected signal is determined for the two optical configurations. Since coincidence of the central frequency (f_0) of the Doppler signal $\{x(t)\}$ and that of the bandpass filter is occasional, the more realistic case of $f_1 \neq f_2$ is examined here.

Case I

When analysing LDV systems with the Mask Optical configuration, the most suitable bandpass filter to employ is the "box car" type, such that the power spectrum of the noise $\{n(t)\}$ in Equation 10 may be expressed as

$$\Phi_n(f) = \frac{N_i}{2\beta} (f_1 - \beta \leq f \leq f_1 + \beta) \quad (11a)$$

with

$$R_n(\tau) = N_i \frac{\sin 2\pi\beta\tau}{2\pi\beta\tau} \cos 2\pi f_1 \tau, \quad (11b)$$

where N_i = Total noise power and 2β = bandwidth of filter.

From Equation 10

$$\Phi_{y1}(f) = \Phi_{x1}(f) + \Phi_n(f) \quad (12a)$$

and

$$R_y(\tau) = R_x(\tau) + R_n(\tau). \quad (12b)$$

A schematic representation of Equation 12a is given in Fig. 5a. Employing Equations 3 and 11, and defining the following relationships.

$$\eta_i = \frac{\psi_i}{N_i} = \text{Input signal-to-noise ratio} \quad (13)$$

$$K_1 = \frac{\beta^2}{A_1^2} = (\text{squared}) \text{ bandwidth ratio} \quad (14)$$

$$\alpha = \frac{f_1^2}{f_0^2} = \text{central frequency ratio}, \quad (15)$$

it may be shown that for the Mask System, the mean zero crossing rate is given by

$$\lambda_1 = 2 \left[\frac{(\eta_i + \alpha)f_0^2 + \frac{A_1^2}{30} (3\eta_i + 10K_1)}{\eta_i + 1} \right]^{\frac{1}{2}}. \quad (16)$$

For $\alpha = 1$, i.e. $f_0 = f_1$

$$\gamma_1 = \frac{1}{A_1^2} \left[\frac{\lambda_1^2}{4} - f_0^2 \right] = \frac{1}{30} \frac{3\eta_i + 10K_1}{\eta_i + 1}, \quad (17)$$

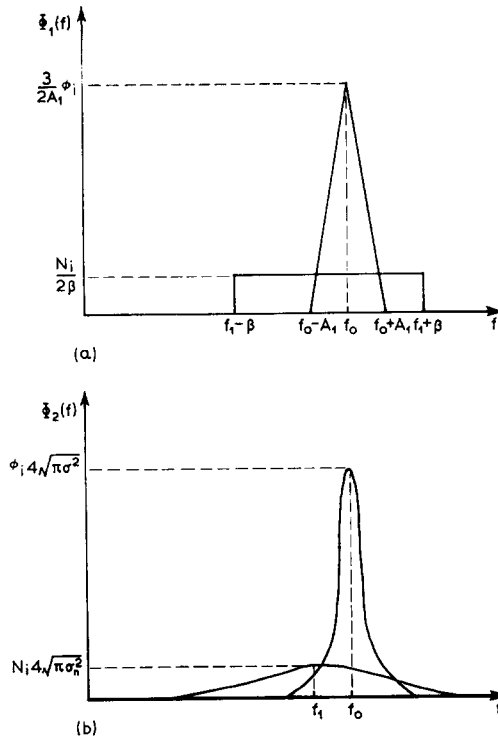


Figure 5 (a) Mask velocimeter system. Power spectrum of "box-car" bandpass filter output for additive input noise on Doppler signal. (b) Gaussian beam velocimeter system. Power spectrum of Gaussian bandpass filter output for additive input noise on Doppler signal.

$$\gamma_1 \rightarrow \frac{1}{10} \text{ as } \eta_i \rightarrow \infty .$$

γ_1 is a measure of the square of the bias (i.e. increase in the zero crossing rate) introduced by the additive noise.

Case II

For the Gaussian beam optical set-up, an appropriate bandpass filter to consider is the Gaussian bell filter, such that the power spectrum of the noise component $\{n(t)\}$ on its output is given by

$$\Phi_n(\omega) = N_i 4\sqrt{\pi\sigma_n^2} \exp - \sigma_n^2 (\omega - \omega_1)^2 \quad 0 \leq \omega \leq \infty \quad (18)$$

where

$$\omega = 2\pi f; \quad \omega_1 = 2\pi f_1 ,$$

with the corresponding autocorrelation function

$$R_n(\tau) = N_i \exp - \frac{\tau^2}{4\sigma_n^2} \cos 2\pi f_1 \tau , \quad (19)$$

where

N_i = Total noise power and $\frac{1}{\sqrt{2\sigma_n^2}} =$ rms bandwidth of the bandpass filter centred at f_1 Hz.

Thus, after bandpass filtering, the power spectrum of the signal obtained from the Gaussian beam set-up is

$$\Phi_{v2}(f) = \Phi_{x2}(f) + \Phi_n(f), \quad (20)$$

where $\Phi_x(f)$ is given by Equation 5, and $\Phi_n(f)$ by Equation 18. An illustration of Equation 20 is given in Fig. 5b. Employing Equations 7a and 20, and the definitions (13) and (15) with the following additional definition

$$K_2 = \frac{\sigma^2}{\sigma_n^2} = \text{bandwidth ratio}, \quad (21)$$

the mean zero crossing rate may be determined as

$$\lambda_2 = 2 \left[\frac{(\eta_i + \alpha)f_0^2 + \frac{A_2^2}{4\pi}(\eta_i + K_2)}{\eta_i + 1} \right]^{\frac{1}{2}} \quad (22a)$$

where $A_2 = \frac{1}{\sqrt{2\pi\sigma^2}}$; and for $\alpha = 1$, i.e. $f_0 = f_1$, this yields

$$\gamma_2 = \frac{1}{A_2^2} \left(\frac{\lambda_2^2}{4} - f_0^2 \right) = \frac{1}{4\pi} \left(\frac{\eta_i + K_2}{\eta_i + 1} \right). \quad (22b)$$

Thus

$$\gamma_2 \rightarrow \frac{1}{4\pi} \text{ as } \eta_i \rightarrow \infty.$$

Notice the similarity between Equations 16 and 21; and 17 and 22. For identical weighting constants, and identical signal-to-noise bandwidth ratios, i.e.

$$A_1 = A_2; K_1 = K_2 = 20 \text{ (say)},$$

Fig. 6 gives a plot of γ_1 and γ_2 versus η_1 (input signal-to-noise power). The values of γ_1 and γ_2 tail off asymptotically to $1/10$ and $1/4\pi$ respectively, for large η_1 , i.e. values given by Equations 9a and b. This figure facilitates comparison of the effective bias in the zero crossing rate introduced by the presence of additive noise on the Doppler signal. It also indicates the relative sensitivity of the Mask system signals to additive noise.

5. Mean rate of drop-out

To determine the so-called “drop-out” rate it is necessary to establish the rate at which the envelope of the bandpass filtered Doppler signal $\{x(t)\}$ of Equations 1 and 2 crosses an arbitrary barrier. Employing Equation 2 as the representation of the Doppler signal (under steady flow conditions) with zero additive noise, it may be shown that $\{x(t)\}$ is

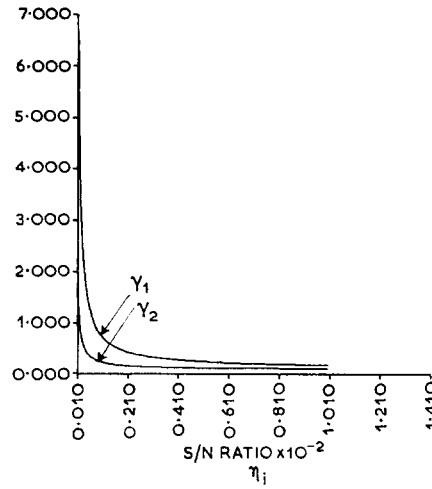


Figure 6 Variation in the relative bias (γ_1 and γ_2 of Equations 17 and 22b in the zero crossing rate of the Doppler signal with additive noise, against input signal-to-noise ratio (η_i).

normally distributed with (say) zero mean and variance ψ_i . Thus the probability density function of $\{x(t)\}$ is

$$p(x) = \frac{1}{\sqrt{2\pi\psi_i}} \exp - \frac{x^2}{2\psi_i} \quad (23)$$

Orthogonality of the components $\{a(t)\}$, $\{b(t)\}$ lead to a Rayleigh distribution for the envelope process $\{H(t)\}$, and a uniform distribution for the phase $\{\phi(t)\}$. Thus

$$p(H) = \frac{H}{\psi_i} \exp - \frac{H^2}{2\psi_i} \quad 0 \leq H(t) \leq \infty \quad (24)$$

$$p(\phi) = \frac{1}{2\pi} \quad 0 \leq \phi(t) \leq 2\pi \quad (25)$$

Let a preset threshold level H_0 indicate signal drop-out. Hence H_0 represents the signal level which cannot be distinguished from zero by the processing hardware. Thus a signal drop-out is registered whenever $\{H(t)\}$ falls below H_0 . This is equivalent to stating that a signal drop-out occurs whenever the phasor in Fig. 1 falls within a circle of radius H_0 centred at the origin.

Considering two time instants $t, t + \tau$, where τ is small, it may be shown that the probability of the envelope process crossing the threshold H_0 in the time interval $(t, t + \tau)$ only once is obtained from

$$P_1(\tau) = [(H(t) - H_0)(H(t + \tau) - H_0) < 0] \quad (26)$$

Defining the normalized autocorrelation of the low-pass equivalent of the process $\{x(t)\}$ as

$$\rho(\tau) = \frac{1}{\psi_i} \int_0^\infty \Phi_x(f) \cos 2\pi(f - f_0) \tau df \quad (27)$$

Then the joint probability density function of $\{H(t)\}$ for the two time instants t , $t + \tau$ is given by

$$p(H_1, H_2) = \frac{H_1 H_2}{\psi_i^2 (1 - \rho^2)} \exp - \frac{H_1^2 + H_2^2}{2\psi_i (1 - \rho^2)} I_0 \left(\frac{\rho H_1 H_2}{\psi_i (1 - \rho^2)} \right) \quad (28)$$

where

$$H_1 = H(t) ; H_2 = H(t + \tau) ; \rho = \rho(\tau)$$

$I_0(x)$ = Modified Bessel function of the first kind and zero order.
 ψ_i is given in Equation 23.

Thus Equation 26 is equivalent to

$$p_1(\tau) = 2 \int_0^{H_0} \int_{H_0}^{\infty} p(H_1, H_2) dH_1 dH_2 \quad (29)$$

The solution of the above equation is unduly complicated; however with the use of Marcum's Q functions [9] this may be expressed as

$$p_1(\tau) = 2 \exp - \frac{H_0^2}{2\psi_i} \left[Q(l, \rho l) - Q(\rho l, l) \right], \quad (30)$$

where

$$l = \frac{H_0}{\psi_i \sqrt{1 - \rho^2(\tau)}},$$

and Marcum's Q function is

$$Q(u, v) = \int_v^{\infty} x \exp - \left(\frac{u^2 + x^2}{2} \right) I_0(ux) dx. \quad (31)$$

Employing asymptotic expansions, Equation 30 may be reduced to

$$p_1(\tau) = 2 \exp - \frac{H_0^2}{2\psi_i} \cdot \operatorname{erf} \left[\frac{l}{\sqrt{2}} (1 - \rho) \right] \quad (32)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

Defining λ_{H_0} as the mean rate of level crossings (at H_0) of the envelope process $\{H(t)\}$ and employing a Taylor's expansion of $p_1(\tau)$ about $\tau = 0$, with the fact that $\rho'(0) = 0$, for a stationary process with a symmetric power spectrum; Equation 32 yields

$$p_1(\tau) = \lambda_{H_0} \tau = 2 \sqrt{-\rho''(0)} \tau \frac{H_0}{\sqrt{2\pi\psi_i}} \exp - \frac{H_0^2}{2\psi_i}. \quad (33)$$

To a first order of approximation, the mean interval between drop-outs is obtained from Equation 33 as

$$T_{H_0} = \frac{2}{\lambda_{H_0}} = \frac{\exp \frac{H_0^2}{2\psi_i}}{H_0} \sqrt{\frac{2\pi\psi_i}{-\rho''(0)}}. \quad (34)$$

Defining

$$\mu^2 = \frac{H_0^2}{\psi_i} \text{ and } T_\mu = T_{H_0} \sqrt{\frac{-\rho''(0)}{\pi}} \quad (35)$$

where μ is the ratio of the threshold level to the rms value of the Doppler signal, $\{x(t)\}$, we obtain a plot of T_μ versus μ in Fig. 7. The relationship tends asymptotically to infinity as $\mu \rightarrow 0$, and as $\mu \rightarrow \infty$. This satisfies the intuitive argument that for a very small threshold level (μ very small), the time interval between drop outs would be very large; theoretically infinite when $\mu = 0$, as no drop-out occurs below this level. While for very large μ (very high threshold level), the Doppler signal is mostly below a very large threshold level, and again the interval between threshold crossings is large.

From the curve of Fig. 7, and from Equation 35, it is shown that the minimum for T_μ occurs at $\mu = 1/\sqrt{2}$, indicating that the highest rate of drop-outs exists when the threshold level $\{H_0\}$ is chosen exactly equal to the rms Doppler signal level ($\sqrt{\psi_i}$).

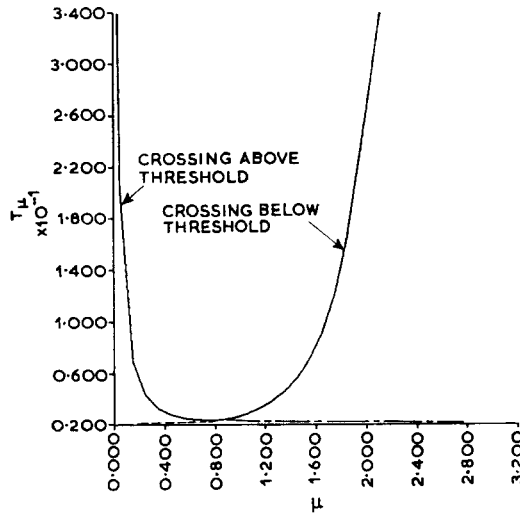


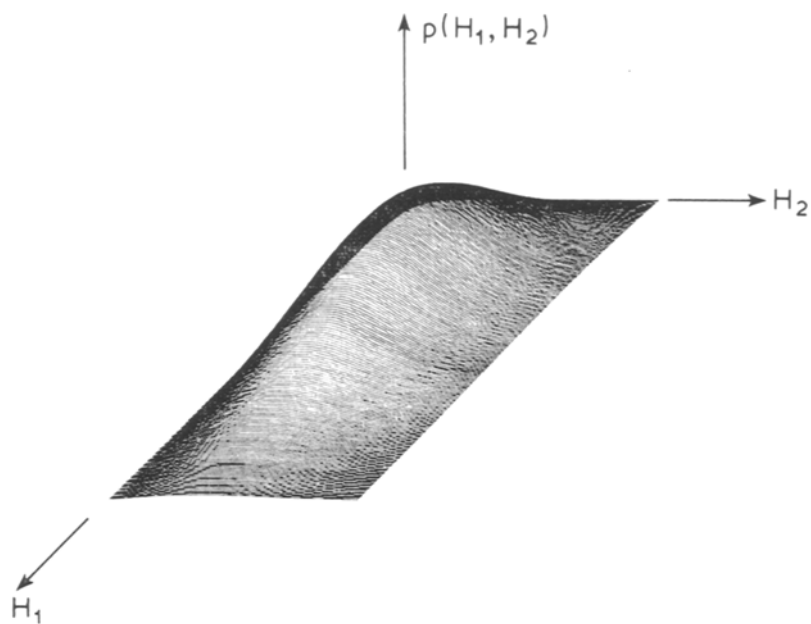
Figure 7 Time interval between level crossings (s) versus normalized threshold level (μ).

5.1. Effect of additive noise on the drop-out rate

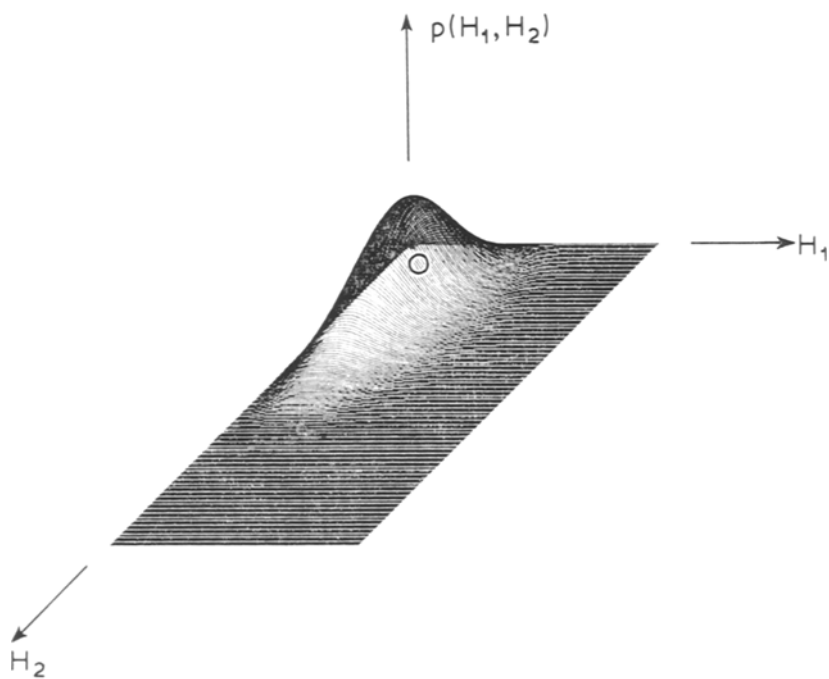
Extending the arguments given in Section 4.2 to the results developed in the previous section, on the assumption of additive white noise to the Doppler signal, being passed through bandpass filters centred at frequency f_0 ($\alpha = 1$ in Equation 15); the mean drop-out rate for the Mask system with a "box-car" filter is given by

$$\lambda_{H_0}(1) = 2\mu \exp(-\mu^2) A_1 \sqrt{\frac{4\pi}{30} \left(\frac{3\eta_i + 10K_1}{\eta_i + 1} \right)}, \quad (36)$$

where A_1 is defined in Equation 3, μ is given by Equation 35, and η_i and K_1 are the same as in Equations 13 and 14 respectively.



(a)



(b)

Figure 8 Schematic for the bivariate probability density function of correlated Rayleigh distributed variables. (a) Autocorrelation function $\rho(\tau) = 0.5$, (b) Autocorrelation function $\rho(\tau) = 0.0$.

Similarly for the Gaussian beam system

$$\lambda_{H_0}(2) = 2\mu \exp(-\mu^2) A_2 \sqrt{\frac{\eta_i + K_2}{\eta_i + 1}}, \quad (37)$$

where A_2 is as given in Equation 6 and Equation 22.

Thus for identical signal-to-noise ratios (η_i), weighting constants ($A_1 = A_2$), and bandwidth ratios ($K_1 = K_2$), the Mask system is marginally more sensitive to the drop-out ratio rate than the Gaussian Beam system. This follows from

$$\text{Lt}_{\eta_i \rightarrow \infty} \frac{\lambda_{H_0}(1)}{\lambda_{H_0}(2)} = \sqrt{\frac{2\pi}{5}} \quad (38)$$

6. Computational results

Figs. 6 and 7 summarize the analysis and results developed in Sections 4 and 5. Figs. 8a and b are 3-D graphic display of Equation 28, for various values of the correlation function (ρ), with $\psi_i = 1$. Notice the smoothing out of the peak by the presence of correlation between the two variables (H_1, H_2). The figures are a novel description of the bivariate distribution of two Rayleigh distributed variables, and to the author's knowledge are the first of their kind. Notice that the projection on the ($H_1 = 0$) plane or ($H_2 = 0$) plane, yields a univariate density function of the Rayleigh kind.

7. An autodyne tracking filter for laser anemometry

In order to introduce the autodyne tracking filter, it will be initially assumed that the signal is the output from a laser anemometer having the following characteristics. Less favourable situations will be considered afterwards. Particle density is high and particle size moderately uniform. The signal-to-noise ratio is good without being ideal and the rate of change of fluid velocity is sufficiently slow that the rate of change df_0/dt of the Doppler frequency f_0 satisfies the relation $df_0/dt \ll (\Delta f_0)^2$, where Δf_0 is the spectral width of the Doppler signal for constant velocity. In this situation the Doppler signal closely resembles the output from a tuned filter for which the input is white noise and the centre frequency and bandwidth of which are equal to f_0 and Δf_0 respectively. Then, as the fluid velocity changes, the effect is as if the centre frequency f_0 of the filter was changed to suit. Assuming for the moment that f_0 is constant, the Doppler signal can be thought of as a sine wave of frequency f and amplitude a , where both a and f fluctuate at the rate Δf_0^{-1} . The rms fluctuation in $f \simeq \Delta f_0$ while the amplitude fluctuates in a Gaussian fashion about zero. If the phasor describing the laser Doppler signal is viewed from a frame of reference that rotates at the frequency f_0 , then the signal phasor V_s will wander at random over the complex plane, subject to a symmetrical Gaussian probability distribution and to a rate limitation such that it takes an average time of Δf_0^{-1} to pass from one side of the distribution to the other. In this situation the frequency fluctuation will be, on average, inversely proportional to amplitude a . If therefore it is required to average out the frequency fluctuations, to obtain the best estimate of the value of f_0 , it is best to weight the average in favour of high values of a . In fact the best average is obtained if the weighting is proportional to a .

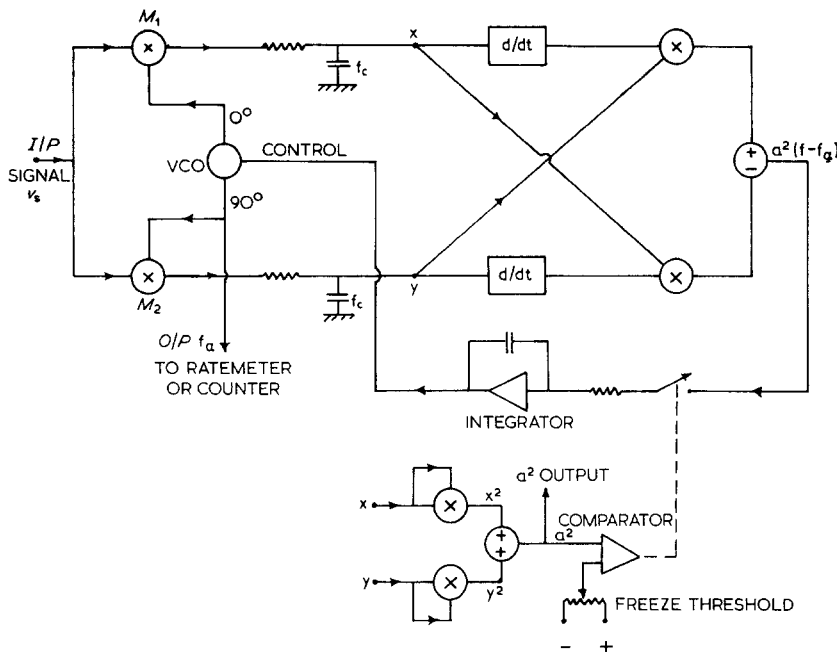


Figure 9 Autodyne tracking filter with "freeze" system.

We obtain the weighted average as follows. An output frequency f_a is required which is equal to $\overline{a^2 f / a^2}$, i.e. $\overline{a^2(f - f_a)} = 0$. This means that the value f_a must be established, continually compared with f , and the difference integrated over the desired averaging period and applied to f_a in such a sense as to make the mean difference between f and f_a zero. The integration constant will also need to be modulated with a^2 so that f_a follows f promptly when a^2 is high but remains nearly constant when a^2 is low.

The system shown in the upper part of Fig. 9 derives the quantity $a^2(f - f_a)$. f_a is the frequency of a two-phase oscillator which is mixed with the Doppler signal in the multipliers M_1 and M_2 . The output from each multiplier will have terms of frequency $f - f_a$ and $f + f_a$. The sum terms are removed by the low-pass filters (cut-off f_c) oscillator. Thus if v_s was a simple sine wave of frequency f , and if the filter outputs were connected to the x and y plates of an oscilloscope, a circle would be described at the frequency $f - f_a$. It should be noted particularly that the sense of rotation reverses when the sign of $f - f_a$ goes negative. If one now uses the relation $a^2 \cdot 2\pi f = x\dot{y} - y\dot{x}$ it is a simple matter to apply analogue processing to derive $a^2(f_0 - f_a)$. If now the oscillator is voltage controllable in frequency, it is simply necessary to apply the signal $a^2(f - f_a)$ to the input of an integrator and to use the integrator output to control the frequency of the oscillator. Thus the system continuously compares f_a with f and applies a correction based on a prior average of $f_a - f$ which is weighted with a^2 .

It is a matter of semantics whether one terms the system a tracking filter or a tracking oscillator. The oscillator frequency is the "output", which can be counted or fed to a

simple ratemeter. The system is a filter in the sense that only input frequencies over the range $f_a - f_c$ to $f_a + f_s$ are accepted. Thus system bandwidth variation is a simple matter. Where signal-to-noise ratio is good, f_c can be made large to minimize the possibility of the system "losing" the signal when velocity fluctuations are fast and extensive. When the signal-to-noise ratio is poorer it becomes necessary to reduce f_c to keep the signal large compared with the noise presented to the final multipliers. An interesting feature of the above system is that its average output is the first moment of the power spectrum of the signal accepted, the moment being about f_a .

We now consider the behaviour when particle density is low and the signal is in the form of isolated bursts. If there was no noise, the system would, in principle, function ideally. That is, the value of f_0 would remain absolutely constant in the absence of a signal and would move towards the frequency of a burst with a response time that is proportional to the (amplitude of the burst)⁻². In this situation it is highly advantageous to know when the bursts occur, and what their amplitude is. The quantity a^2 is given by $x^2 + y^2$, so that another pair of multipliers and an adder provide the signal.

Where noise is present, and the combined amplitude is below the average noise level, it is then almost certain that the signal is absent. In that situation one requires no correction, so that the system ignores the noise between bursts. The most appropriate technique is to compare the a^2 signal with a threshold level that is just above that of the noise, using a comparator, and to use the comparator output to activate a switch to open circuit the input to the integrator. Thus the original a^2 weighting is overridden by this "freeze" signal. The facility is also useful even in the absence of noise to combat slight d.c. offsets in the outputs from the multipliers.

Several systems of the above type have been built. One system, operating at a Doppler frequency of 100 KHz gave a response time of 30 μ s, corresponding to a frequency response of 5 KHz. Another, incomplete, system gives a response time of 10 μ s for a Doppler frequency of 1 MHz. These parameters were all measured using a simulated Doppler signal having the same amplitude and frequency fluctuations as the true Doppler signal, but with the ability to modulate the frequency by an external signal to simulate velocity changes. The noise response characteristics are such that the effect of added noise is almost negligible when the spectral density of the white noise is less than that of the signal, but catastrophic when the white noise density is greater than that of the signal. The first condition also requires that the filter bandwidth is sufficiently narrow that the level of total noise passing through the filter is less than that of the signal. The above comments are confirmed both in theory and practice. A 10 dB ratio between noise and the unmodulated signal gives a negligible increase over the normal frequency fluctuations of the signal when the signal level is the greater. For noise densities greater than 10 dB above the signal, tracking is impossible. For a ratio of 10 dB, tracking is possible when bandwidth and correction loop gain are reduced to give a slowing of 10^4 , i.e. to a response time of 300 ms. A slowing of (noise/signal power)⁴ can be justified theoretically.

8. Conclusions

Two types of simulation methods have been considered, and it is found that for given

signal bandwidth, the analogue system is restrictive, while the digital approach allows signals arising from various configurations of the velocimeter to be closely simulated. However, frequency modulation effects are difficult to implement on the digital computer.

The method of instantaneous Doppler signal frequency estimation by zero crossing techniques has been examined, and it is shown that the bias introduced due to the finite (ambiguity) bandwidth of the Doppler signal is proportional to the weighting constants (A_1 and A_2) and thus to the bandwidth/centre frequency ratio. However, additive noise introduces factors which depend upon the signal-to-noise ratios and signal bandwidths/filter bandwidth ratio, but these are independent of the individual signal power or noise power. The zero crossing rate of signals arising in the Mask system is marginally more sensitive to additive noise, than that in the Gaussian beam set-up.

The ever present "drop-outs" in the Doppler signal have been studied and their mean rate has been determined. This again depends upon the optical configuration, and under first order approximations, the interval between drop-outs has been calculated in terms of the threshold level of signal detection.

The autodyne tracking system has been shown to yield very accurate estimates of the instantaneous Doppler frequency. The effect of weighted averaging of the output tracking frequency involves feedback with memory which stabilizes the loop dynamics. The system is shown to function effectively even under poor signal-to-noise ratios.

Acknowledgements

C.G. gratefully acknowledges the financial assistance of the Science Research Council.

References

1. T. H. WILMSHURST, *J. Phys. E.: Sci. Instrum.* **4** (1971) 77-80.
2. E. R. PIKE, D. A. JACKSON, P. J. BOURKE, and D. I. PAGE, *ibid* **1** (1968) 727-730.
3. C. GREATER and T. S. DURRANI, *ibid* **4** (1971) 24-26.
4. R. J. ADRIAN, *ibid* **5** (1972) 91-95.
5. W. D. T. DAVIES, *Wireless World*, Aug. (1966) 407-412.
6. S. O. RICE, *Bell Syst. Tech. J.* **23** (1944) 282-322; **24** (1945) 46-156.
7. A. PAPOULIS, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill (1965).
8. T. S. DURRANI, *Trans. IEEE on Communications*, **COM-20** (1972) 296-307.
9. J. I. MARCUM, *Tables of the Q-Function*, Rand Corp. Res. Memo RM-399, January 1950.